# HYDROSTATICS

THE ALLAHABAD LAW JOURNAL PRESS, ALLAHABAD
PRINTER—J. K. SHARMA

PUBLISHERS
KITAB - MAHAL
56/A, ZERO ROAD, ALLAHABAD

# HYDROSTATICS

A TEXT-BOOK FOR THE USE

OF

B.A. AND B.Sc. STUDENTS

OF

INDIAN UNIVERSITIES

*By*

**B. N. PRASAD**

DR. ES SC. (PARIS), PH.D. (LIVERPOOL), M.SC., F.N.I.

*University of Allahabad*

KITAB-MAHAL

ALLAHABAD

1943

## PREFACE

The necessity of a suitable text-book on *Hydrostatics* to meet the requirements of the B. A. and B. Sc. students of Indian Universities has long been felt. While some of the existing books on the subject are too advanced, others are below the standard, dealing with experimental parts in excessive detail and avoiding the elegant methods of the *Calculus*. The aim of this book is to supply this want.

The book has been planned so as to cover the syllabus of the B. A. and B. Sc. Examinations of all Indian Universities. The topics common to the various syllabuses have been dealt with in the main body of the book. Two topics, namely 'the determination of specific gravity,' and 'the stability of floating bodies,' whose inclusion in the prescribed syllabuses is not so universal, have been treated in the Appendices A and B respectively. My experience of teaching during the last twenty years has given me the impression that undergraduate students feel more at home with a text-book which does not contain here and there matter which may be considered superfluous for the preparation of the examination in view. The arrangement of the matter, as given in this book, will enable students belonging to most of the Universities to have the subject-matter of the prescribed syllabus treated in a continuous manner, from which they are not expected to omit anything substantial for the purpose of a thorough preparation for their examination.

Every care has been taken to present the matter in a clear and lucid style and to establish the propositions in

a perfectly logical manner. But keeping in view the class of readers for whom this book is intended, it has not been found advisable to sacrifice simplicity and directness for the sake of rigour and excessive refinements. Since the subject of the *Calculus* is everywhere included in the B. A. and B. Sc. courses, no hesitation has been felt in making use of it whenever it was thought that with its help the results could be obtained more easily or expressed more concisely. This being a text-book, intended for class-use, several alternative proofs of some of the important theorems have also been given. It is believed that the historical and biographical notes occasionally given will be found interesting.

To facilitate the use of this text-book the subject-matter of the various sections is indicated by side-headings in bold type, in which also are printed prominently important results and propositions. This plan will prove, it is hoped, particularly useful to students in their revision work. The book is profusely illustrated, there being more than a hundred diagrams in it. For ready reference and convenience, a list of important results relating to the mensuration of solids, centre of gravity and radius of gyration is given in Appendix C.

The book contains an ample number of solved examples, carefully selected to illustrate all important varieties of problems. The large number of examples included in the various exercises are well-graded. Of these some have been constructed, many have been selected from the Cambridge Mathematical Tripos, the examinations of the various Indian Universities and the Public Service examinations; the rest are such as are common to practically all the standard books on the subject. In case of harder examples, hints for their solution are given.

I am greatly indebted to my friends and former pupils Messrs. Nawa Nath Misra, M.A., P.C.S., and Nirvikar Saran, M.SC., Lecturer in Mathematics at C. C. College, Cawnpore. While the former has sorted and verified most of the examples, the latter, besides verifying the remaining examples, has given me the benefit of general consultation. My thanks are also due to my pupil Mr. Chandrika Prasad, M.SC., who has verified some examples included in the Miscellaneous set My colleague Dr. Gorakh Prasad, D.SC., has all along helped me with his valuable criticism, suggestions and advice, and has gone through the proofs of the whole of the book. I am sincerely grateful to him for his generous assistance.

Inspite of careful printing and reading there may have remained some residual errors. All corrections and suggestions for improving the utility of the book will be gratefully received.

*University of Allahabad*
*November*, 1943

B. N. PRASAD

## CONTENTS

# HYDROSTATICS

## CHAPTER I

## FLUIDS AND FLUID PRESSURE

1·1. **Different States of Matter.** Matter or substance in nature may exist in the solid form as ice, iron, wood, and stone, or in the fluid form as water, oil, air and steam. If we take any portion of a solid body, we observe that it has a definite size and a definite shape which cannot be changed except by the application of a certain amount of force. **Fluids,** on the other hand are substances which *flow or are capable of flowing.* If we take a given portion of a fluid substance, such as water or air, we find that it possesses no definite shape and that it moulds itself to the form of the vessel in which it is contained.

Fluids again are of two kinds, **Liquids** and **Gases.** A liquid, like water or oil, is a fluid which is incompressible, or very nearly so. Even with the application of an enormous amount of force, the *volume* of a given quantity of liquid can be changed only to a very slight extent, although any force, however small, would easily change its *shape.* On the other hand, a gas, like air, is a fluid which is compressible, and a given portion of it can be made to expand indefinitely by increasing sufficiently the space to which it has access. Neither the *volume* nor the *shape* of a given portion of a gas is fixed, its volume and shape being always the same as that of the vessel in which it is contained.

1·11. The distinguishing characteristics of the three kinds of substances or states of matter, **Solid, Liquid, and Gas** may be roughly expressed as follows :—

*A solid has a definite size and a definite shape.*

*A liquid has a definite size but not a definite shape.*

*A gas has neither a definite size nor a definite shape.*

1·12. By suitable devices and changes of temperature etc., a substance can be made to pass from one of these three states to another. By increasing the temperature, a solid piece of ice can be melted into the liquid state as water, and the water again can be evaporated into the gaseous state as steam. By increasing the temperature sufficiently, a metal in the solid state can be melted and by the the application of more intense heat, it can be turned into gaseous state. A gas, like air, oxygen etc., can be reduced to a liquid state, and afterwards to the solid state.

1·13. **Hydrostatics** *is the branch of Mathematics which deals with the conditions of equilibrium of masses of fluids and of solids in contact with them.*

1·14. The word Hydrostatics is derived from a Greek word meaning the Statics of Water. Archimedes (c. 250 B.C.) who discovered the method of testing the purity of metal by weighing them in and outside water, and extended those principles to the equilibrium of floating bodies like ship etc., is considered to be the father of this science. He dealt with this subject in his work—Περὶ ὀχουμένων, which, though now lost, is still preserved in its Latin version by Guillaume de Moerbeck (1269), known as "*De iis quae vehuntur in humido.*" This has been now translated into French by Adrien Legrand and is entitled "*Le traite des corps flottants d' Archimède,*" 1891.

The principles and applications of pneumatic machines were developed by Ctesibius of Alexandria, his pupil Hero (B. C. 120), and by Vitruvius, Frontinus (A. D. 100) and others. Substantial advance in the theory of Hydraulics was made by Torricelli (1643), the inventor of the barometer.

In the writings of Stevinus of Bruges (c. 1600 A. D.) several basic theorems of this science may be found explained and enunciat-

ed; but it is generally accepted that the modern exact theory of Hydrostatics is due to Blaise Pascal (1623-1662) who in his two treatises "*Traité de l' équilibre des liqueurs*" and "*Traité de la pesanteur de la masse de l'air*", clearly enunciated and illustrated for the first time the fundamental principles of the subject.

The elastic properties of gases were investigated and developed among others by Boyle, Mariotte, Charles and Gay Lussac. The fundamental principles having been thus established, the subject was subsequently refined and completed by Newton, Cotes, Bernoulli, d'Alembert and other mathematicians of the 18th century.

**1·15. Perfect Fluids.** We have seen that fluids are of two kinds, liquids and gases. There is a certain property which is common to most of these substances whether they be liquids or gases. This property consists in the mobility of their particles and in the ease with which portions can be separated from masses of fluids. If a thin plate or lamina be immersed edgeways through water, the resistance to its motion in the direction of its plane, i.e., force of the nature of friction, is so small that for all practical purposes it may be considered to be negligible. This leads us to suppose the existence of a fluid which is absolutely incapable of exerting any tangential action of the nature of friction between its different portions.

In nature, however, no such fluids are to be found which are wholly devoid of exerting tangential resistance. If water in a cup be set revolving, we know that after some time it will come to rest. This is due to the frictional resistance between the water and the cup and between different portions of water, for had this tangential resistance been entirely absent, the water would go on revolving and would not come to rest. But just as from the existence of nearly rigid bodies in nature, we are led to suppose the existence of perfectly rigid bodies and from the behaviour of nearly smooth bodies we postulate the exis-

tence of perfectly smooth bodies, similarly from the extreme smallness of tangential action offered by fluids, we postulate the existence of a hypothetical fluid which is wholly incapable of exercising any tangential force and, therefore, the only action that it can exert will be along the normal to the surface Such an ideal fluid will be called a perfect fluid and its fundamental property may be stated as follows:—

*A* **Perfect Fluid** *is a substance which is incapable of exerting any tangential force; the only direction along which it can exert an action will be along the normal to the surface with which it is in contact.*

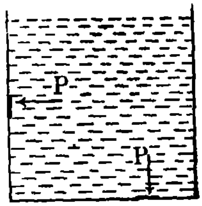
The postulation of a perfect fluid greatly simplifies the mathematical discussions connected with the phenomena of fluids. Throughout this book we shall be dealing with fluids which are perfect in the above sense.

**1·16. Viscous Fluids.** As has been remarked already, no fluid in nature is perfect. Some fluids like water and alcohol change almost instantaneously their shape under the action of a force, while there are some like treacle, honey and tar which offer appreciable resistance to forces tending to alter their shape and for which time is necessary before an appreciable change of shape occurs. If we take two glasses of the same shape and size filled with water and treacle and pour their contents on two large flat dishes, it will be observed that water pours out much more quickly than treacle. Again, the poured out water will adapt itself at once to the shape of the dish, but the treacle will first appear in a sort of heap which will gradually get flattened out and then cover the dish ultimately. Fluids in which internal forces of the nature of friction come into play to retard the relative motion of their particles are called *viscous fluids.* But even a very viscous

fluid will ultimately yield to a force, however small, if that force acts for a sufficiently long time.

It must be noted that viscosity manifests itself only when the fluid is in motion. In the case of a mass of fluid at rest, viscosity does not come into effect.

**1·2. Fluid Pressure.** Suppose a vessel with a horizontal base and vertical sides, is full of fluid, say water. If a hole be made in the base or in the side, and the hole covered with a plate which fits it exactly, we know that the plate will remain in position at rest only when some external force is applied to it. The fluid must, therefore, be exerting some force on the plate. If $P$ be the




amount of the force necessary to be applied to counterbalance the action of the fluid, then $P$ is the measure of the **fluid pressure** *on the whole area of the plate* or **the fluid thrust** *on that area.*

In order to obtain the fluid pressure over an area round a point R *within the fluid,* suppose a rigid plate of area $\alpha$ placed so as to contain the point R. Now imagine that all the fluid is removed from one side of the plate and then a force of $Q$ units is required to be applied to keep the plate at rest. Then $Q$ would be the measure of the fluid pressure over an area $\alpha$ round the point R within the fluid.

If the fluid pressure be the same for each equal element of the area, as it would be when the elements of area are taken over the horizontal base of the vessel, the pressure on the area is said to be *uniform.* On the other hand, if the fluid thrust is not the same on different equal portions of area, as, for instance, on the vertical side of the vessel, the pressure is said to be *varying* or *non-uniform* over that area.

**1·21. Pressure at a point.** If the fluid pressure be uniform over an area, the term **pressure at a point** is used to express the pressure per unit area. Thus if $P$ be the measure of the uniform pressure over the area $A$, then $P/A$ stands for the *pressure at a point* over that area.

In case the pressure be varying, we may consider a small element of area $\alpha$ enclosing a point. *If $p$ be the fluid pressure over this area $\alpha$, then the limit of $p/\alpha$, as $\alpha$ tends to zero, is defined to be the pressure at that point.*

Evidently this mode of defining the pressure at a point is applicable to the first case also when the pressure is uniform, and consequently the above may be taken to be the general definition of the term *pressure at a point.*

**1·211. Mean Pressure.** The above expression $p/\alpha$ is termed the **mean** or **average pressure,** so that the *mean pressure on a given plane area is the uniform pressure which would produce the same resultant thrust on that area as the actual total pressure.*

The idea of the *pressure at a point* may now be expressed in terms of the *mean pressure* as follows:—

*The* **pressure at a point** *of an area is the limit of the mean pressure on a small area enclosing the point as the area is diminished indefinitely.*

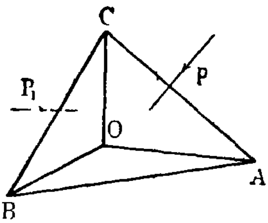
If $p$ denotes the pressure at a point, $dA$ a small element of area surrounding that point and $p+dp$ the mean pressure over the area $dA$, where $dp$ tends to zero with $dA$, then the fluid pressure over the element of area $dA$ is given by $(p+dp)dA$. This may be taken to be $pdA$, correct to small quantities of first order.

**1·22. Equality of Pressure in different directions.** Making use of the fundamental property that the pressure of a fluid is always normal to the surface with

which it is in contact, the following proposition will be now proved :—

**The pressure at a point of a fluid at rest is the same in all directions.**

Let $O$ be any point in the fluid. Taking $OA$, $OB$, $OC$ mutually at right angles, consider the mass of the fluid in the shape of a small tetrahedron $OABC$ with three mutually perpendicular faces $OBC$, $OCA$ and $OAB$.




Let $P_1$, $P$ denote the mean pressures across the faces $OBC$, $ABC$ and $p_1$, $p$ the pressure at the point $O$ normal to these two faces respectively. Let $\theta$ be the angle between $ABC$ and $BOC$, i.e., between $OA$ and the perpendicular to $ABC$. The triangle $BOC$ being the projection of the triangle $ABC$ on the plane $BOC$, we have

$$\triangle BOC = \triangle ABC \cos\theta. \qquad . \quad . \quad . \quad (1)$$

Supposing the fluid is subject to an external field of force* whose effect can be measured as so much force per unit mass, the forces acting on the mass of the fluid in the tetrahedron are the following :—

(1) The resultant of the system of external forces acting on the fluid. If gravity be the only external force, then

* Those who have studied Physics must be familiar with the phenomenon that a magnet exerts on other magnets in the space around it a force of attraction or repulsion which varies inversely as the square of the distance from the magnet. Again, there may exist a centre of attraction, which exerts on all particles of matter around it a force proportional, say, to the cube of the distance, or in fact, a force governed by any other law.

Such forces which are present in a certain region are said to constitute what is known as a *field of force*.

There may be various types of sources to produce a field of force. Gravity itself is an example of a field of force in which each particle is attracted towards the centre of the earth with a force proportional to the mass of the particle.

this will be the weight of the tetrahedron of fluid. In every case this resultant being proportional to the mass of the fluid, may be taken proportional to its volume. Let the component parallel to $OA$ of this resultant of external forces be represented by

$$K \,.\, \text{volume } OABC = K.\tfrac{1}{6}\, OA \,.\, OB \,.\, OC,$$
$$= K \,.\, \tfrac{1}{3}\, OA \,.\, \triangle OBC.$$

(2) The fluid thrust on the face $OBC$ which is $P_1 \,.\, \triangle OBC$ along $OA$.

(3) The thrust on the face $ABC$ which is $P \,.\, \triangle ABC$ acting normal to the face $ABC$.

(4) The thrusts on the faces $AOB$ and $AOC$ which are both normal to $OA$.

The mass of the tetrahedron of fluid being in equilibrium, by resolving parallel to $OA$ we get

$$P_1.\triangle OBC - P.\triangle ABC.\cos\theta + K.\tfrac{1}{3}\, OA.\triangle OBC = 0,$$

or $$(P_1 - P).\triangle OBC + \tfrac{1}{3}\, K.OA.\triangle OBC = 0,$$

or $$P_1 - P + \tfrac{1}{3}\, K.OA = 0 \quad . \quad . \quad . \quad (2)$$

Now let $OA$, $OB$, $OC$ diminish indefinitely so that the tetrahedron dwindles into the point $O$ and the mean pressures $P_1$ and $P$ become pressures $p_1$ and $p$ respectively at the point $O$. Applying this limiting process, we, therefore, get from (2) that $p_1 = p$, that is, the pressure at $O$ along $OA$ is equal to the pressure at $O$ perpendicular to $ABC$.

In a similar manner it can be shown that $p$ is equal to the pressure at $O$ along $OB$ or $OC$. By varying the relative magnitudes of $OA$, $OB$ and $OC$, it can be shown that this equality of pressure will continue to hold whatever be the direction of the plane $ABC$, i.e., whatever be the direction in which $p$ is taken at $O$. This proves the proposition.

**1·23. Dimensions of pressure.** Pressure at a point being force per unit area, its dimensions in terms of the

fundamental units of mass, length and time are

$$MLT^{-2}/L^2 = ML^{-1}T^{-2}.$$

1·24. **Transmissibility of Liquid Pressure.** An important principle, known as Pascal's law, may now be stated as follows:—

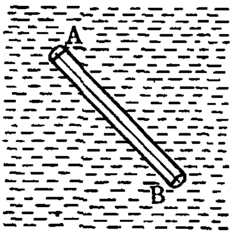
**An increase of pressure at any point of a liquid at rest under given external forces is transmitted without change to every other point of the liquid.**

The truth of this principle is usually demonstrated experimentally, but a theoretical proof also can be given as explained below.

Let $A$, $B$ be any two points in the liquid.

Case I. *When the straight line joining $A$, $B$ lies entirely in the liquid.*

About the straight line $AB$ as axis construct a cylinder of small cross-section $\alpha$ with plane ends at right angles to $AB$. This cylinder of liquid is in equilibrium under the following forces:—




(1) The thrusts $p\alpha$ and $p'\alpha$ on the plane ends at $A$ and $B$ where $p$ and $p'$ denote the pressures at $A$ and $B$ respectively.

(2) The thrusts on the elements of the curved surface of the cylinder which must be perpendicular to $AB$.

(3) The resultant external force on the liquid whose component along $AB$ may be represented by, say, $\sigma$.

Now resolving these forces along $AB$, we have

$$p\alpha - p'\alpha + \sigma = 0,$$

or

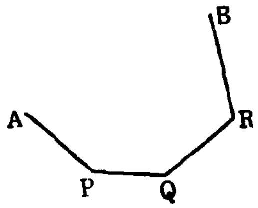
$$p' - p = \frac{\sigma}{\alpha} = \text{a constant.} \quad . \quad . \quad (\gamma)$$

This shows that any increase of pressure at $A$ must be accompanied by an equal increase of pressure at $B$, for then only the equation ($\gamma$) will continue to hold true.

Case II. *When the straight line joining $A$, $B$ is not entirely in the liquid.*

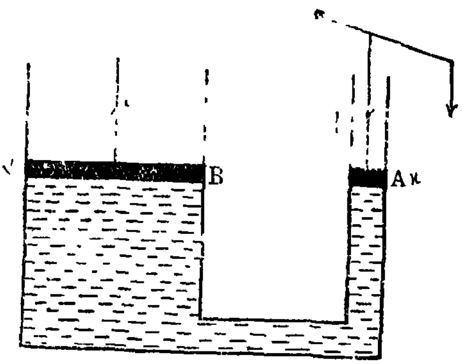
Join $A$, $B$ by means of a series of straight lines $AP$, $PQ$, $QR$, $RB$, each of which lies entirely in the liquid.

Now by the case I, we have the additional pressure at $A$ = the additional pressure at $P$ = the additional pressure at $Q$ = the additional pressure at $R$ = the additional pressure at $B$.




**1·25. Bramah's or Hydraulic Press.** The foregoing principle of the transmissibility of pressure is utilised in a very useful mechanical contrivance called a hydraulic press or Bramah's press. The object of this machine is to get a comparatively small force to manifest itself into a largely multiplied one.

Essentially the apparatus consists of two vertical cylinders communicating with each other near their bases; one of the cylinders is of a much wider cross-section than the other. The cylinders are closely fitted with water-tight pistons $A$ and $B$, the cross-sections of which are, say, $x$ and $X$ respectively.




Suppose a downward force $p$ is applied by the piston

$A$. This will create an additional pressure $p/x$ per unit area which, by Pascal's law, will be transmitted to every part of the liquid. Consequently the resulting upward thrust on the second piston will be

$$P = Xp/x,$$

which shows that the thrusts on the two pistons are proportional to their cross-sections.

We see, therefore, that by suitably adjusting the ratio of the cross-sections of the two cylinders, any force, however small, may be made to support any weight, however large. This is sometimes described as a '*hydrostatic paradox*.' We must note, however, that in practice this multiplication is limited by the fact that the sides of the vessel cannot be strong enough to support an indefinitely great amount of pressure that might be put upon them.

**1·251. Example.** *In a Bramah's press the diameter of the larger piston is* 20 *inches and that of the smaller one is* 2·5 *inches. What force must be applied to the smaller piston to raise a weight of* 1000 *tons at the other piston?*

Let $P$ lbs. wt. be the required force to be applied to the smaller piston. The pressure induced per sq. inch is

$$P/\pi(1\cdot25)^2 \text{ lbs. wt.}$$

The larger piston can lift a weight

$$= \pi(10)^2 \,.\, P/\pi(1\cdot25)^2 \text{ lbs. wt.}$$

$\therefore$ $$100P/(1\cdot25)^2 = 1000\times2240,$$

or $$P = 15\cdot625 \text{ tons weight.}$$

**1·3. Density.** The masses of equal volumes of different substances may be compared in terms of density which is defined as follows:—

*The* **Density** *of a homogeneous substance is the mass of unit volume of that substance.*

A substance is said to be *homogeneous* if equal volumes taken from any part of it have always equal masses, otherwise the substance is called *heterogeneous*.

The mass of a cubic foot of water at temperature 4°C. (when a given quantity of water occupies the least volume) is approximately 62 425 lbs. For rough calculations the density of water may be taken to be 1000 oz. or 62 5 lb. per cubic foot. The gramme having been chosen to represent the mass of a cubic centimetre of water at 4°C., the density of water is 1 in the metric system.

**1·31. Weight in terms of Density.** *If $W$ be the weight of a given substance in poundals, $\rho$ its density in lbs. per cubic foot, $V$ its volume in cubic feet, and $g$ the acceleration due to gravity in foot-second units, then*

$$W = g\rho V. \qquad \ldots\ldots \qquad (1)$$

For, if $M$ be the mass of the substance, we know that

$$W = Mg.$$

Now, since

$$\begin{aligned} M &= \text{mass of } V \text{ cubic feet of substance,} \\ &= V \times \text{mass of one cubic foot,} \\ &= V\rho, \end{aligned}$$

we have

$$W = g\rho V.$$

A similar result will be true if the units be taken in C. G. S. system.

**1·4. Specific Gravity.** Quite often we are not concerned with the actual masses of unit volumes of substances but only with the *relative* masses or the relative weights of equal volumes of two substances. In order to express this relation the term *specific gravity* (or written in a shortened form sp. gr.) is introduced whose formal definition is given below.

*The* **Specific Gravity** *of a substance is the ratio of the weight of any volume of that substance to the weight of an equal volume of some standard substance.*

For the sp. gr. of liquids the standard substance is usually taken to be pure water at 4°C. temperature, whereas for gases, air is generally used as the standard substance.

Since the weights of substances are proportional to their masses, it follows that the *specific gravity of a substance is the ratio of its density to that of the standard substance.* For this reason the specific gravity of a substance is sometimes called its "*relative weight*" or its "*relative density.*"

It must be noted that density being mass per unit volume, its dimensions are expressed by $ML^{-3}$ whereas specific gravity, since it denotes ratio of two weights, has no dimensions; it is simply a number.

**1·41. Weight in terms of Specific Gravity.** *If $W$ denotes the weight of a volume $V$ of a substance whose specific gravity is $s$,* and *$w$ be the weight of a unit volume of the standard substance, then*

$$W = V\,s.w \quad \ldots\ldots \quad (1)$$

Since $s = \dfrac{\text{wt. of a unit volume of the substance}}{\text{wt. of a unit volume of the standard substance}}$

$\therefore$ wt. of a unit volume of the substance $= s.w.$

$\therefore$ wt. of $V$ units of volume of the substance $= V.s.w.$

Hence, $\quad W = V.s.w.$

**Note.** The term "intrinsic weight" or "specific weight" is sometimes used for $s.w$, that is, for the weight of a unit volume of the substance.

**1·5. Specific Gravity of Mixtures.**

(1) *To find the specific gravity of a mixture of different substances of given volumes and specific gravities.*

Let $V_1$, $V_2$, $V_3$,......be the volumes of the different substances, and $s_1$, $s_2$, $s_3$,......their specific gravities. Then assuming the volume of the mixture to be the sum of the

volumes of its constituents; the weight of the mixture by 1·41 (1) is

$$(V_1 + V_2 + V_3 + \ldots\ldots)\bar{s}w, \quad . \quad . \quad . \quad (1)$$

where $s$ denotes the specific gravity of the mixture and $w$ the weight of a unit volume of the standard substance. But this must be equal to the sum of the weights of the constituents of the mixture, viz.,

$$V_1 s_1 w + V_2 s_2 w + V_3 s_3 w + \ldots\ldots \quad . \quad . \quad (2)$$

Now equating (1) and (2) and cancelling the factor $w$, we get

$$\bar{s} = \frac{V_1 s_1 + V_2 s_2 + V_3 s_3 + \ldots\ldots}{V_1 + V_2 + V_3 + \ldots\ldots} \quad . \quad . \quad . \quad (3)$$

In case the mixture has a volume $V'$ different from the sum of the volumes of its constituents, then the corresponding formula for the sp. gr of the mixture is given by

$$\bar{s} = \frac{V_1 s_1 + V_2 s_2 + V_3 s_3 + \ldots\ldots}{V'} \quad . \quad . \quad . \quad (4)$$

(II) *To find the specific gravity of a mixture of different substances of given weights and specific gravities.*

Let $W_1$, $W_2$,......be the weights of the different substances, $s_1$, $s_2$, .....their specific gravities, and $w$ the weight of a unit volume of the standard substance. Then by 1·41 (1), their respective volumes will be

$$\frac{W_1}{s_1 w}, \frac{W_2}{s_2 w}, \ldots\ldots .$$

If the volume of the mixture is the sum of the volumes of the component parts and if $\bar{s}$ denotes the sp. gr. of the mixture, then the weight of the mixture is

$$\left(\frac{W_1}{s_1 w} + \frac{W_2}{s_2 w} + \ldots\ldots \right)\bar{s}w. \quad . \quad . \quad . \quad (5)$$

This must be equal to the sum of the weights of the substances of which the mixture is made, viz.,

$$W_1 + W_2 + \ldots . \qquad \ldots \quad (6)$$

Equating (5) and (6), we get

$$s = \frac{W_1 + W_2 + \ldots\ldots}{W_1/s_1 + W_2/s_2 + \ldots\ldots} \quad \ldots \quad (7)$$

The above formula can be suitably modified if there is any change in the total volume when the mixture is made.

**1·51. Density of Mixtures.** If the densities instead of the specific gravities of the different substances are given, formulae analogous to those established in 1·5 can be easily obtained for the density of the mixture.

**1·6. Illustrative Examples.** (i) *If a volume of* 10 *cm. of a liquid of density* 0·8 *grammes per c. cm. be mixed with* 15 *c. cm. of a liquid of density* 0·6 *grammes per c.cm., find the density of the mixture.*

Let $\sigma$ be the density of the mixture. Then

$$(10 + 15)\sigma = (10 \times 0·8) + (15 \times 0·6),$$

or $$25\sigma = 17,$$

or $$\sigma = \frac{17}{25} = 0·68 \text{ (grammes per c.cm.)}.$$

(ii) 10 *lbs. weight of a liquid of specific gravity* 1·25 *is mixed with* 6 *lbs. weight of a liquid of sp. gr.* 1·15. *What is the sp. gr. of the mixture?* [Calcutta, 1937]

If $w$ be the weight of a cubic foot of water, the volumes of the two liquids are

$$\frac{10}{1·25 \times w} \text{ c. ft. and } \frac{6}{1·15 \times w} \text{ c. ft.}$$

respectively.

Now if $s$ represents the sp. gr. of the mixture, we have

$$\left(\frac{10}{1·25 \times w} + \frac{6}{1·15 \times w}\right) sw = \text{total weight}$$
$$= 10 + 6,$$

or $$\left(8 + \frac{120}{23}\right) s = 16,$$

or $$\left(1 + \frac{15}{23}\right) s = 2,$$

or $$s = \frac{2 \times 23}{38} = \frac{23}{19} = 1\ 2105.$$

(iii) *An alloy of zinc of sp. gr.* 7 2 *and copper of sp. gr.* 8 95 *has a mass of* 467 *grammes. Its volume is* 60 *c cm. Find the volume of each component.*

Let $v_1$ c.cm. and $v_2$ c.cm. be the volumes of the zinc and copper respectively. Then since the sp. gr. of a substance is given by the ratio of the mass of a unit volume of the substance to the mass of a unit volume of water, the masses of copper and zinc are $7\ 2 \times v_1$ grammes and $8 \cdot 95 \times v_2$ grammes respectively.

We have now
$$v_1 + v_2 = 60,$$
$$7\ 2v_1 + 8\ 95v_2 = 467.$$

Hence, solving these equations,
$$1\ 75\ v_1 = 70,$$
or $$v_1 = 40 \text{ c. cm.},$$
and $$v_2 = 20 \text{ c. cm}$$

## Examples I

1. In a Bramah's press the diameters of the larger and smaller pistons are 50 cm. and 4 cm. respectively. Find the mass which can be supported by a kilogram placed on the smaller piston.

2. In a Bramah's press the piston can safely bear a pressure of 1200 lbs. wt. per sq. foot. What will be the greatest weight that can be placed on the smaller piston, its cross-section being 5 sq. inch?

3. A cylindrical pipe which is filled with water opens into another pipe, the diameter of which is three times its own diameter; if a force of 20 lbs. wt. be applied to the water in the smaller pipe, find the force on the open end of the larger pipe, which is necessary to keep the water at rest.

4. If the sections of the cylinders of a Bramah press be 18 square inches and 1 sq. foot respectively, what pressure must be applied to the smaller cylinder to produce a pressure of two tons upon the larger?

5. In a Bramah's press a total pressure of 1 ton is produced when a pressure is applied to the smaller piston by means of a force of 5 lbs. placed at the end of the lever. If the diameters of the pistons are as 8 : 1, find the ratio of the arms of the lever employed to work the piston.

6. A vessel full of water is fitted with a tight cork. How is it that a slight blow on the cork may be sufficient to break the vessel? [*U. P. C. S.* 1940]

7. The sp. gr. of cork being 0·24, find the volume of water that weighs as much as a cubic yard of cork.

8. Find the sp. gr. of an alloy of gold and copper made in the ratio 5 : 2, their sp. gr. being 19·4 and 8·84 respectively.

9. Two metals of which the specific gravities are 11·22 and 7·25, when mixed in certain proportions without condensation form an alloy whose sp. gr. is 8·72; find the proportion by volume of the metals in the alloy.

10. Two volumes of sp. gravities $s$ and $s'$ and of volumes $v$ and $v'$, having been mixed, the sp. gr. of the mixture is found to be $\sigma$. Find the volume of the mixture. [*M. T.*]

11. Three pints of a liquid whose sp. gr. is 8 are mixed with 5 pints of another liquid whose sp. gr. is 1·04. Find the sp. gr. of the mixture if there is a contraction of 5 per cent on the joint volume.

12. When equal volumes of two substances are mixed, the sp. gr. of the mixture is 4; when equal weights of the same substances are mixed, the sp. gr. of the mixture is 3. Find the sp. gravities of the substances. [*Calcutta*, 1938]

CHAPTER II

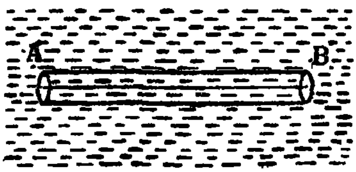
# THEOREMS RELATING TO PRESSURE, MEASURE OF PRESSURE AND WHOLE PRESSURE

**2·1. Fluid at rest under gravity.** A mass of fluid may be kept in equilibrium by a variable field of force. In this and succeeding chapters, however, we shall be mainly concerned with the special case when the field of force is uniform and is produced by gravity alone. Hence, *unless stated to the contrary, we shall assume that the only external force acting on the mass of fluid to be considered is its weight due to gravity.*

When we want to specify that even gravity is not acting on a body or liquid, we call it a *light* body or liquid. When, on the other hand, we want to make it pointedly clear that gravity *is* acting, we say that the body or liquid is *heavy*.

**2·2. Equality of Pressure at all points in a horizontal plane.** *The pressure in a mass of fluid at rest under gravity is the same at any two points in the same horizontal plane.*

Let $A$, $B$ be any two points in the same horizontal plane. Join $A$, $B$ and let us suppose that the straight line $AB$ lies entirely in the fluid.




About $AB$ as axis let a cylinder of small cross-section be constructed with its plane ends perpendicular to $AB$.

The only forces acting on the cylinder parallel to $AB$ are the two thrusts at the plane ends $A$ and $B$; for the other forces, viz., the thrust on the curved surface and the weight of the cylinder of fluid are perpendicular to $AB$. The cylinder of fluid being in equilibrium, the thrusts at plane ends $A$ and $B$ are equal. Since the areas of plane ends at $A$ and $B$ are equal, the mean pressure at $A$ is equal to the mean pressure at $B$. Now supposing the cross-section of the cylinder to diminish indefinitely, we get

$$\text{pressure at } A = \text{pressure at } B.$$

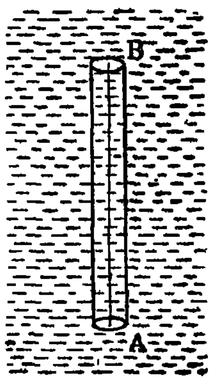
In **2·32** it will be shown that the theorem is true even if the straight line $AB$ does *not* lie entirely in the fluid.

**2·3. Pressure in heavy homogeneous liquid.** *In a homogeneous fluid at rest under the action of gravity, the difference between the pressures at two points varies as the difference of the vertical depths of these two points.*

Let $A$, $B$ be any two points, and let us consider two cases.

Case I. When the straight line $AB$ is vertical and lies entirely in the fluid.

About $AB$ as axis describe a cylinder of small cross-section $\alpha$ with plane horizontal ends at $A$ and $B$. Let $p$ and $p'$ denote the pressures at $A$ and $B$ respectively, $h$ the vertical distance between $A$ and $B$ and $w$ the weight of unit volume of the fluid.



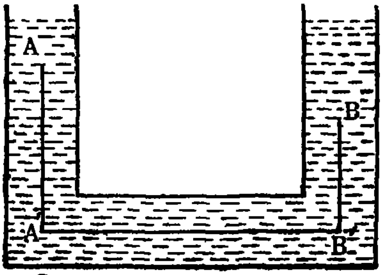

The vertical forces acting on this cylinder of fluid are its weight $h\alpha w$ acting downwards, the thrusts of the surrounding fluid at its plane ends which are $p\alpha$ acting upwards at $A$ and

$p'\alpha$ acting downwards at $B$. Since the cylinder of fluid is in equilibrium, we get on resolving vertically

$$p\alpha - p'\alpha = h\alpha w,$$

or $$\mathbf{p - p' = hw}. \qquad . \quad . \quad . \quad . \quad . \quad (1)$$

Case II. When the straight line $AB$ is not vertical and does not lie entirely in the fluid.




Let the points $A$, $B$ be situated in the fluid as shown in the figure.

Let $A'$, $B'$ be two points in a horizontal line vertically below $A$, $B$. Now, by case I,

$$\text{pressure at } A' - \text{pressure at } A = AA'.w,$$

and $$\text{pressure at } B' - \text{pressure at } B = BB'.w.$$

But $A'$ and $B'$ being in the same horizontal line, the pressures at $A'$ and $B'$ are equal. Hence

$$\text{pressure at } B - \text{pressure at } A = (AA' - BB')\, w,$$

which varies as the difference of depths between $B$ and $A$.

The proposition is, therefore, completely proved.

**2·31. Difference of pressure in terms of weight.** Since $hw$ represents the weight of a cylinder of fluid of unit cross section and of height equal to the vertical distance between $A$ and $B$, the foregoing proposition may be also stated as follows :—

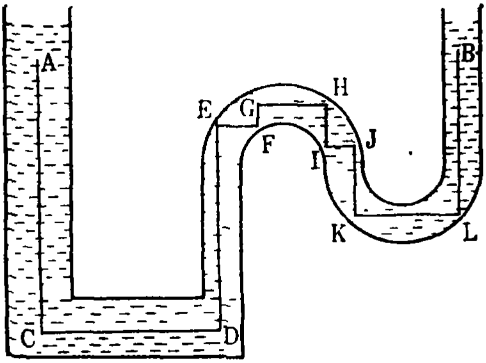
*The difference of pressure between any two points in a fluid is equal to the weight of a cylinder of fluid of unit cross section, and of height equal to the distance between those two points.*

It may be noted that the above is true even if the fluid be not homogeneous.

**2·32. Extension of the theorem of 2·2.** It can now be easily seen from the foregoing theorem that the pressures

at two points $A$ and $B$ in the same horizontal plane are the same even when the straight line joining $A$, $B$ does not lie entirely in the fluid. For, if $A$ and $B$ are in the same plane, the difference between their depths being zero, the horizontal pressure at $A$ is equal to the pressure at $B$.

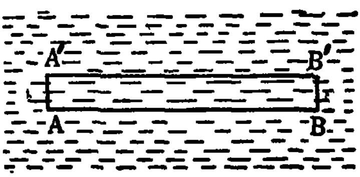
Similarly it can be demonstrated that the proposition is true for any two points in the same level even when to join them, as is shown in the adjoined figure, a series of straight lines is required each of which lies entirely in the fluid.




*Thus for a fluid at rest under gravity, horizontal planes are* **surfaces of equal pressure.**

**2·4. Surfaces of equal density.** *In a fluid at rest under gravity, the densities at any two points in the same horizontal plane are equal.*

Let $A$, $B$ be two points in the same horizontal plane and $A'$, $B'$ be two points vertically above at a very short distance, so that $AA' = BB'$.




The pressures at $A$ and $B$ being equal, let each be denoted by $p$; similarly, let the pressures at $A'$ and $B'$ be each $p'$. Let the mean values of the densities between $A$ and $A'$ and between $B$ and $B'$ be denoted by $\rho_1$ and $\rho_2$ respectively, so that the limiting values of $\rho_1$ and $\rho_2$ represent the

densities, at $A$ and $B$ respectively when $AA'$ and $BB'$ are indefinitely diminished.

Now by 2·3 (1), we have

$$p-p' = g\rho_1.AA',$$

and

$$p-p' = g\rho_2.BB'.$$

Hence

$$g\rho_1.AA' = g\rho_2.BB',$$

and since $AA' = BB'$, we get

$$\rho_1 = \rho_2,$$

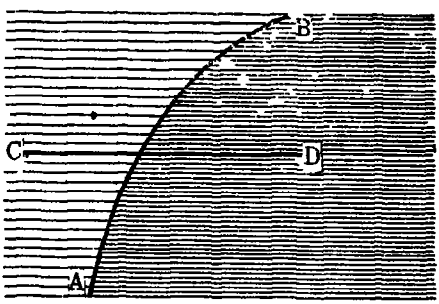
which proves the proposition.

Thus we find that *in a fluid at rest under gravity horizontal planes are* **surfaces of equal pressure and equal density.**

**2·5. Surface of separation a horizontal plane.** From the foregoing proposition the following can be deduced:—

*The surface of separation of two fluids of different densities, which do not mix and are at rest under gravity, is a horizontal plane.*

Let $AB$ represent the surface of separation of two liquids which are at rest under gravity and do not mix. If $AB$ be not horizontal, let it be intersected by a horizontal line $CD$. Then $C$ and $D$ being in the two different liquids, the densities at $C$ and $D$ will be different; but this will be contradictory to the proposition of 2·4. Hence, the surface of separation $AB$ is horizontal.



**2·51. Free surface of a liquid.** As a particular case of the above, let us consider a liquid in contact with atmospheric air assuming that both the liquid and the air are at rest. Then it follows that *the free surface of the liquid which is the surface of separation of the two fluids, must be horizontal.*

*Note.* In the preceding proofs it has been assumed that the weights at different points of the fluid act vertically downwards in parallel directions. This will not be so if the surface of the fluid be large and comparable with the earth in size. Accordingly, the theorem just deduced will not be applicable to the free surface of the sea, which may not be plane.

**2·52. Water finds its own level.** The law that the free surface of a liquid at rest is horizontal is often popularly expressed by saying that "*water finds its own level.*" The implication is that when two vessels containing the same liquid, but of different depths, are put in communication with each other, the liquids in the two vessels readjust themselves in such a manner that the free surfaces of the liquids in the two vessels form parts of the same horizontal plane. It follows similarly that if a homogeneous liquid at rest under gravity has a number of isolated surfaces in contact with the same atmosphere at rest, the free surfaces must all lie in the same horizontal plane.

This property of liquids is utilized in various ways. Upon this depends the possibility of supplying water to a town by means of closed pipes from a water reservoir constructed at an elevation. The water communicated through pipes will always rise to the level of the surface in the reservoir if free to do so.

**2·53. A characteristic of fluids.** It is characteristic of fluids that when two fluids which do not mix, remain in contact in stable equilibrium, the fluid of lower density

occupies the upper position while that of greater density comes lower. In case there be more than two fluids which do not mix, then the position of the strata of these various fluids from the top will be in the order of their increasing densities.

**2·6. Measure of Pressure.** We have seen that the free surface of a liquid exposed to the earth's atmosphere is horizontal. The atmosphere produces on this exposed surface a pressure which, though slightly varying, may be roughly taken to be 14·5 lbs. per square inch.

In 2·3, if $B$ be taken to be a point in the free surface of the liquid and if $\Pi$ denote the pressure due to the atmosphere at $B$, then from 2·3 (1), we have for pressure at $A$

$$\mathbf{p} = \Pi + \mathbf{hw}. \quad \ldots\ldots \quad (1)$$

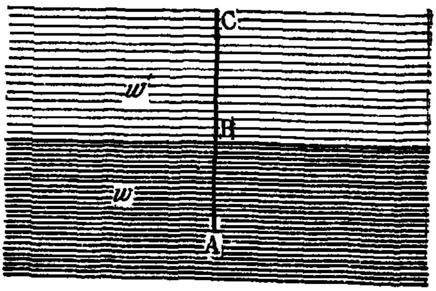
If there is no atmospheric pressure, or if the atmospheric pressure be neglected, then

$$\mathbf{p} = \mathbf{hw}. \quad \ldots\ldots \quad (2)$$

**2·61. Pressure at a point in the lowest layer of different fluids.** Suppose we have a number of layers of different fluids which do not mix and we want to find the pressure at a point situated in the lowest of them.

For the sake of simplicity, let there be two liquids, the weight of unit volume of the lower and upper liquids being $w$ and $w'$ respectively.

Suppose $A$ is a point in the lower liquid. From $A$ draw a vertical line $ABC$ such that $B$ is a point in the common surface of the two liquids and $C$ in the free surface of the upper liquid exposed to the atmosphere. Let



$AB = h$ and $BC = h'$, and suppose $p$ and $p'$ denote the pressures at $A$ and $B$ respectively and $\Pi$ the atmospheric pressure at $C$.

Now, by **2·3** (1), we have

$$p = p' + hw.$$

But since from **2·6** (1), $p' = \Pi + h'w'$, we get

$$\mathbf{p = \Pi + h'w' + hw.} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Proceeding in a similar manner the result (1) can be extended to the case when over the point $A$ there is a number of layers of different liquids which do not mix. Proceeding from the free surface to the point $A$, if there be $n$ liquids of intrinsic weights $w_1, w_2, \ldots\ldots w_n$, and of depths $h_1, h_2, \ldots\ldots h_n$ respectively, then the pressure $p$ at the point $A$ is given by

$$\mathbf{p = \Pi + h_1w_1 + h_2w_2 + \ldots\ldots + h_nw_n.} \quad . \quad . \quad (2)$$

If there be no atmospheric pressure, then the result (2) will become

$$\mathbf{p = h_1w_1 + h_2w_2 + \ldots\ldots + h_nw_n.} \quad . \quad . \quad . \quad (3)$$

The above results are obviously true also for gases, provided there is no mixing among the various gases. But gases generally mix with comparative ease on account of diffusion.

**2·62. Effective Surface.** We have seen in **2·6** (1) that the pressure at a depth $h$ of a liquid whose free surface is exposed to the atmospheric pressure is given by

$$p = \Pi + hw, \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where $\Pi$ denotes the atmospheric pressure and $w$ the weight of unit volume of the liquid or its intrinsic weight. If we imagine the atmosphere to be removed and a stratum of the same liquid of thickness $\Pi/w\ (= h')$ to be placed above the original liquid, then the pressure at a depth $h$ in the

liquid is given by $(h' + h)w$ which is the same as (1). This may be regarded as pressure at a depth $h + h'$ of the liquid, the atmospheric pressure being zero.

*The upper surface of the supposed superimposed liquid of thickness $h'$ is termed the* **Effective Surface,** *or* **surface of zero pressure.**

We see, therefore, that the pressure at any point in a liquid is proportional to its depth below the effective surface.

**2·63. Head of liquid.** When we say that the pressure at a point is due to a **head** of $k$ feet of liquid, the meaning is that the pressure at that point is the same as if the effective surface were $k$ feet above it.

The atmospheric pressure is roughly taken to be that due to a *head* of 34 feet of water or 30 inches of mercury.

**2·7. Illustrative Examples.** (i) *Find the depth in water at which the pressure per sq. inch is* 140 *lbs., assuming the atmospheric pressure to be* 15 *lbs. per sq. inch and the weight of one cubic foot of water* 1000 *ozs.*

The weight of 1 c. ft. of water being 1000 ozs, the weight of 1 c. inch is

$$\frac{1000}{16} \times \frac{1}{12 \times 12 \times 12} \text{ lb.}$$

Now, if $x$ ft. be the depth of water where the pressure is equal to 140 lbs. per sq. inch, we have from 2·6 (1)

$$15 + 12x \times \frac{1000}{12 \times 12 \times 12 \times 16} = 140,$$

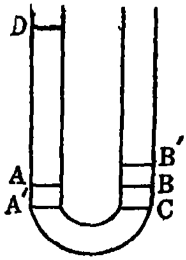
or

$$x = \frac{125 \times 12 \times 12 \times 16}{1000} \text{ ft.}$$

$$= 288 \text{ ft.}$$

(ii) *In a U-tube of uniform cross-section there is some mercury; water is then poured into one limb and it occupies a length of* 9 *inches. If the sp. gr. of mercury be taken to be* 13·5, *find the distance through which the mercury level in the other limb is raised.*

Let $A$, $B$ denote the initial levels of mercury in the two limbs of the tubs; they must be in the same horizontal plane. When water has been poured in, suppose the mercury level in the left limb sinks from $A$ to $A'$ and that in the right limb is raised from $B$ to $B'$. The tube being of uniform corss-section, $AA'$ must be equal to $BB'$. Suppose $DA'$ is the length occupied by water. In the right limb let $C$ be a point in the same horizontal level as $A'$.




If $\Pi$ represents the atmospheric pressure, since the pressures at $A'$ and $C$, being in the same horizontal plane, are equal, we get

$$\Pi+DA'\times w=\Pi+B'C\times 13\cdot5w,$$

or

$$DA'=13\cdot5\times B'C.$$

But since $BB'=AA'=BC$, we have $B'C=2BB'$, and $DA'=9$ inches. Hence

$$9=13\cdot5\times 2BB',$$

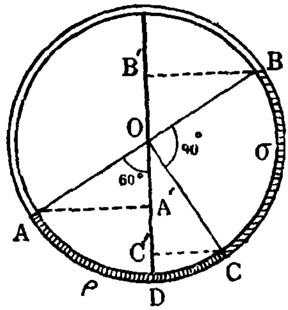
or

$$Bb'=\tfrac{1}{3}\text{ inch.}$$

(iii) *In a uniform circular tube two liquids are placed so as to subtend* $90°$ *each at the centre. If the diameter joining the two free surfaces be inclined at* $60°$ *to the vertical, prove that the densities of the two liquids are as*

$$\sqrt{3}+1:\sqrt{3}-1.$$

Let $A$ and $B$ be the surfaces of the two liquids of densities $\rho$ and $\sigma$ respectively and $C$ their meeting point, so that $\angle AOC=\angle BOC=90°$. The diameter $AOB$ makes with the vertical an angle $AOD=60°$.




Draw $BB'$, $AA'$, $CC'$ perpendiculars to the vertical diameter, of which $D$ is the lowest point.

The pressure at the point $D$ due to the liquid in $AD$ is by **2·62** (2)

$$A'D.\rho g. \qquad \ldots\ldots \quad (1)$$

Also the pressure at the point $D$ when considered due to the two liquids in $DB$, is given by **2·61** (3)

$$C'D.\rho g+C'B'.\sigma g. \qquad \ldots\ldots \quad (2)$$

But since the pressure at $D$ is the same whether considered from the left or from the right side, we get on equating (1) and (2)

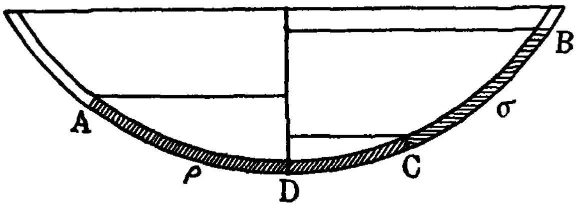
$$A'D.\rho g=C'D.\rho g+C'B'.\sigma g,$$

or $$\varrho . A'C' = \sigma(OB' + OC'),$$
or $$\varrho(OC' - OA') = \sigma(OC' + OB'),$$
or $$\varrho(\cos 30^\circ - \cos 60^\circ) = \sigma(\cos 30^\circ + \cos 60^\circ),$$
or $$\frac{\varrho}{\sigma} = \frac{\sqrt{3}/2 + 1/2}{\sqrt{3}/2 - 1/2}$$
$$= \frac{\sqrt{3} + 1}{\sqrt{3} - 1}.$$

(iv) *A cycloidal uniform tube contains equal weights of two liquids, occupying lengths a and b; if it be placed with its axis vertical, prove that the heights of the free surfaces of the fluids above the vertex of the tube are as*

$$(3a+b)^2 \text{ to } (3b+a)^2. \qquad [M.\ T.]$$

Let the two liquids of densities $\varrho$ and $\sigma$ respectively extend from




$C$ to $A$ and from $C$ to $B$, so that arc $CA = a$ and arc $CB = b$. The axis of the cycloid being vertical, let $D$ be its lowest point.

Let the vertical heights of $A$, $C$ and $B$ be $y_1, y_2$ and $y_3$ respectively. Suppose the arc $DC$ is of length $s$.

Since the weights of the two liquids are equal, we have

$$a\varrho = b\sigma. \qquad \ldots\ldots (1)$$

Equating the pressures at $D$ from the left and right sides, we get

$$y_1 \varrho g = y_2 \varrho g + (y_3 - y_2)\sigma g,$$

or $$\frac{\sigma}{\varrho} = \frac{y_1 - y_2}{y_3 - y_2}. \qquad \ldots\ldots (2)$$

In a cycloid we have the geometrical relation

$$s^2 = 8ry,$$

where $r$ is the radius of the generating circle of the cycloid, $y$ the vertical height above the vertex of a point whose arcual distance from the vertex is $s$. Hence

$$(a - s)^2 = 8ry_1, \qquad \ldots\ldots (3)$$
$$s^2 = 8ry_2, \qquad \ldots\ldots (4)$$
$$(s + b)^2 = 8ry_3. \qquad \ldots\ldots (5)$$

Subtracting (4) from (3) and (5), we get respectively

$$a^2 - 2as = 8r(y_1 - y_2); \; b^2 + 2bs = 8r(y_3 - y_2).$$

$$\therefore \qquad \frac{a^2 - 2as}{b^2 + 2bs} = \frac{y_1 - y_2}{y_3 - y_2} = \frac{\sigma}{\varrho} = \frac{a}{b}, \text{ from (2) and (1),}$$

or $$b(a^2 - 2as) = a(b^2 + 2bs),$$

or $$4abs = ba^2 - ab^2,$$

or $$s = \frac{a - b}{4}. \qquad \ldots\ldots \quad (6)$$

Now from (3) and (5), we get

$$\frac{y_1}{y_3} = \frac{(a - s)^2}{(b + s)^2}$$

$$= \frac{\{a - (a - b)/4\}^2}{\{b + (a - b)/4\}^2} \text{ from (6),}$$

or $$\frac{y_1}{y_3} = \frac{(3a + b)^2}{(3b + a)^2}.$$

## Examples II

1. The sp. gr. of mercury is 13·6. At what depth in mercury will the pressure be equal to that at 500 meters in water?

2. If the pressure of the atmosphere be taken to be 15 lbs. weight per sq. inch, and the weight of water of 1 cu. ft. volume 62·5 lbs. weight, find the pressure per sq. inch at depths of (i) 10 inches, (ii) 20 feet, under the surface.

3. The pressure in a waterpipe at the base of a building is 39 lbs. wt. per sq. inch and on the roof it is 19 lbs. wt. per sq. inch; find the height of the roof. (1 cu. ft. of water weighs 62·5 lbs.)

4. The atmospheric pressure at the surface of a lake is 15 lbs. wt. per sq. inch. Find at what depth will the pressure be 45 lbs. wt. per sq. inch, the weight of a cubic foot of water being taken to be 1000 ozs.

5. The pressure at the bottom of a well is four times that at a depth of 2 feet. What is the depth of the well if the pressure of the atmosphere be equivalent to that of 30 feet of water? [*Benares*, 1938]

6. Two liquids $A$ and $B$ do not mix and have different densities. When $A$ is poured in a vessel to a vertical height $h$, the pressure on the bottom is the same as when $A$ stands to a height $h_1$

and $B$ to a height $h_2$ above it. Show that the ratio of the density of $A$ to that of $B$ is

$$\frac{h_2}{h - h_1}.$$ [*Benares*, 1940]

7. Prove that the pressure at the centre of gravity of a triangular lamina wholly immersed in a homogeneous liquid, in any manner, is one-third the sum of the pressures at the angular points.

8. Prove that if a parallelogram be immersed in any manner in a heavy homogeneous liquid, the sum of the pressures at the extremities of one diagonal is equal to the sum of the pressures at the extremities of the other diagonal. [*Calcutta*, 1936]

9. A layer of mercury 25 cm. deep, (sp. gr. 13·6) is covered by one of water of the same depth; above this there is a layer 50 cm. in depth of oil, (sp. gr. 0·9); find the pressure at the bottom (i) in grammes wt. per sq. cm. (ii) in centimetres of mercury, assuming the atmospheric pressure to be due to a column of 76 cm. of mercury.

10. If there be $n$ fluids arranged in strata of equal thickness, and the density of the uppermost be $\rho$, of the next $2\rho$ and so on, that of the last being $n\rho$; find the pressure at the lowest point of the $n$th stratum and thence prove that the pressure at any point within a fluid whose density varies as the depth is proportional to the square of the depth. [*M. T.*]

11. If a liquid be heterogeneous and of density $\frac{\rho z}{a}$ at a depth $z$, show that the pressure is $\Pi + \frac{g\rho z^2}{2a}$, where $\Pi$ is the atmospheric pressure.

12. The lower ends of two vertical tubes whose cross-sections are 2 and 0·2 sq. inches respectively are connected by a tube. The tubes contain mercury (sp. gr. 13·6). How much water must be poured in the larger tube to raise the level of the mercury in the smaller tube by 2 inches?

13. If $\rho$, $\rho'$ be the densities of two fluids ($\rho < \rho'$) and the lengths of the arms of a U-tube in which they meet be $m$ and $n$ inches respectively, prove that in order that the tube may be completely filled, the height of the column of the lighter fluid above the horizontal plane in which they meet, must be

$$\frac{\rho'(m - n)}{\rho' - \rho} \text{ inches.}$$ [*M. T.*]

14. A small uniform tube is bent into the form of a circle whose plane is vertical. Equal quantities of two fluids of densities $\rho$ and $\sigma$ fill half the tube. Show that the radius passing through the common surface makes with the vertical an angle $\theta$ given by

$$\tan\theta = \frac{\rho - \sigma}{\rho + \sigma}.$$ [*Bombay*, 1940]

15. A fine circular tube in a vertical plane contains a column of liquid, of density $\delta$, which subtends a right angle at the centre, and a column of density $\delta'$ subtending an angle $\alpha$. Prove that the radius through the common surface makes with the vertical an angle

$$\tan^{-1}\frac{\delta - \delta' + \delta'\cos\alpha}{\delta + \delta'\sin\alpha}.$$ [*Agra*, 1932]

16. In the lower half of a uniform circular tube, one quadrant is occupied by a liquid of density $2\rho$, and the other by two liquids of densities $\rho$ and $3\rho$. Prove that the volume of the lower of the two latter liquids is twice that of the other two. [*Lucknow*, 1938]

17. Three fluids whose densities are in A. P. fill a semi-circular tube whose bounding diameter is horizontal. Prove that the depth of one of the common surfaces is double that of the other.

[*Calcutta*, 1916]

18. A tube in the form of a parabola held with its vertex downwards and axis vertical, is filled with two different liquids of densities $\delta$ and $\delta'$. If the distances of the free surface of the liquids from the focus be $r$ and $r'$ respectively, show that the distance of their common surface from the focus is

$$\frac{r\delta - r'\delta'}{\delta - \delta'}.$$ [*Lucknow*, 1929; *Agra*, 1930]

19. A closed tube in the form of an equilateral triangle contains equal volumes of three liquids which do not mix and is placed with its lowest side horizontal. Prove that, if the densities of the liquids are in A.P., their surface of separation will be at points of trisection of the sides of the triangle. [*M. T.*]

**2·8. Thrust on a surface.** When a surface is in contact with a liquid, the liquid exerts thrusts normal to the surface at every element of the area. If the surface be a plane surface, then the direction of the thrust being normal to the plane area, is everywhere the same. Thus the

thrusts on all the elements of the plane surface form a system of parallel forces, which on being compounded will give the *resultant* or *total thrust* of the liquid on the plane surface. If the surface be a curved surface, the thrusts at various elements of the area will not be in the same direction, and hence if they be simply added up together, their mere arithmetical sum will not give the resultant thrust. But whether the surface be plane or curved, this arithmetical sum of the thrusts at all the elements of the area is termed the **whole pressure**, which may be defined as follows:—

*The* **Whole Pressure** *of a fluid on any surface with which it is in contact, is the sum of all the normal thrusts exerted by the fluid on every element of the surface.*

The formula arrived at in the following section for the calculation of the *whole pressure* for a plane surface is equally applicable, both in method and expression, also to the case of a curved surface. But since for a curved surface, the *whole pressure* has no mechanical significance whatsoever, its calculation does not serve any practical purpose.

It should be well noted that in the case of a *plane* surface, the Whole Pressure and the Resultant Thrust are the same, but *not so* for a *curved* surface. The determination of the resultant thrust for a curved surface will be discussed in the next chapter.

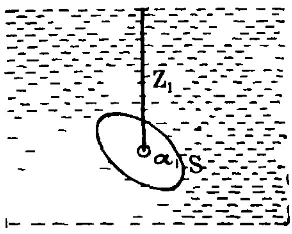
**2·81. Whole Pressure on a plane surface.** Suppose a mass of homogeneous liquid is in contact with a plane surface. If the atmospheric pressure is neglected, then the free horizontal surface of the liquid will be the surface of zero pressure. When the atmospheric pressure is not neglected, the surface of zero pressure will be the *effective surface*. We can establish now the following useful theorem:—

**If an area S of a plane surface be in contact with a homogeneous liquid, the whole pressure or resultant thrust of the liquid upon the surface is given by**

$$w.\,S.\,\bar{z},$$

**where $\bar{z}$ represents the depth of the centre of gravity of the area S below the surface of zero pressure and w the weight of unit volume of the liquid.**

Let the area $S$ be divided into an indefinitely large number of small elements of area $\alpha_1, \alpha_2, \alpha_3, \ldots\ldots$ and let the depths of these elements of area below the surface of zero pressure be $z_1, z_2, z_3, \ldots\ldots$




Since the pressure at a depth $z_1$ is $z_1 w$, the thrust on the small element of area $\alpha_1$, may be taken to be $w\alpha_1 z_1$. Similarly the thrusts on $\alpha_2, \alpha_3, \ldots\ldots$ are given by $w\alpha_2 z_2$, $w\alpha_3 z_3, \ldots\ldots$ Hence the whole pressure on the area $S$

$$= w\alpha_1 z_1 + w\alpha_2 z_2 + w\alpha_3 z_3 + \ldots\ldots$$

$$= w\Sigma\,\alpha_1 z_1. \qquad \ldots\ldots \quad (1)$$

But we know from *Statics* that if $\bar{z}$ represents the distance of the centre of gravity of $S$, then

$$\bar{z} = \frac{\Sigma\,\alpha_1 z_1}{\Sigma\,\alpha_1} = \frac{\Sigma\,\alpha_1 z_1}{S}.$$

Putting this value of $\Sigma\,\alpha_1 z_1$ in (1), we get the whole pressure or resultant thrust on the area $S$

$$= wS\bar{z}.$$

**2·82. Other forms of expressing whole pressure.** The resultant thrust $wS\bar{z}$ obtained above is sometimes called the **total thrust** or **total pressure** of the liquid on the surface $S$. Quite often when there is no chance of confusion, even the word 'total' is dropped, and by saying

'pressure on an area,' we understand total pressure on that area.

Obviously the whole pressure $wSz$ may also be expressed in the following ways:—

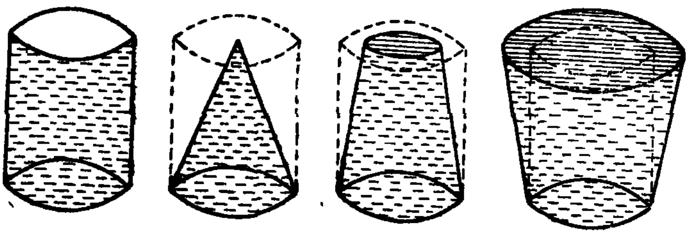
(i) *The whole pressure of a liquid on a surface is equal to the weight of a column of liquid of which the base is equal to the area of the surface, and the height is equal to the depth of the centre of gravity of the surface below the surface of zero pressure.*

(ii) *The whole pressure of a liquid on a plane area is equal to the product of the area and the pressure at the centre of gravity of the area.*

We see from above that *the mean or average pressure throughout the area is the pressure at the centre of gravity of the plane area.*

**2·83. Whole Pressure on a horizontal base.** Since the pressure at all points in a horizontal plane is the same, *the thrust on a horizontal surface will be given by the product of the pressure at any point of the surface and the area of the surface.*

It follows that the thrust on the horizontal base of a



vessel containing liquid does not depend upon the shape of the vessel, nor upon the quantity of liquid contained; it depends only upon the depth of the liquid and the area of the horizontal base. Thus in the above figure the four vessels (shown in elevation) have got different shapes and contain different amounts of the same liquid, but the area of their bases and the depth of the liquid being the

same in all the four, they will have the same resultant thrust on their bases, which will be equal to the weight of the liquid contained in the cylinder-shape vessel.

**2·9. Illustrative Examples.** (i) *Calculate the total thrust on one side of a rectangular vertical dockgate 45 feet wide, immersed in sea-water to a depth of 30 feet, given that 1 c. ft. of sea-water weighs 1025 ozs.*

*If there is fresh water on the other side of the gate, find its depth so that the resultant thrusts on the two sides are equal.* [*Madras*, 1934]

The area immersed under sea-water

$$= 45 \times 30 \text{ sq. ft.}$$

The depth of the C.G. of the immersed area

$$= 15 \text{ ft.}$$

$\therefore$ the total thrust on the dock-gate

$$= 45 \times 30 \times 15 \times \frac{1025}{16} \text{ lbs.}$$
$$= 1297265 \cdot 625 \text{ lbs.}$$

If $d$ be the depth of the fresh water on the other side, the resultant thrust on the other side

$$= (45 \times d) \times \frac{d}{2} \times \frac{1000}{16} \text{ lbs.}$$

The resultant thrusts on the two sides being equal, we have

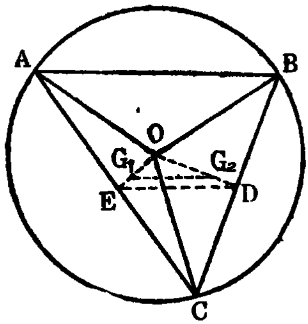
$$45 \times d \times \frac{d}{2} \times \frac{1000}{16} = 45 \times 30 \times 15 \times \frac{1025}{16},$$

or $$\frac{1000d^2}{2} = 30 \times 15 \times 1025,$$

or $$d^2 = 30^2 \times \frac{1025}{1000},$$

or $$d = 3\sqrt{102 \cdot 5} \text{ ft.},$$
$$= 30 \cdot 372 \text{ ft. nearly.}$$

(ii) *A triangle $ABC$ is immersed in a liquid, its plane being vertical and the side $AB$ in the surface; if $O$ be the centre of the circumscribed circle of the $\triangle ABC$, prove that the pressure on the $\triangle OCA$ : the pressure on the $\triangle OCB$ :: $\sin 2B$ : $\sin 2A$.*




Let $D$ and $E$ be the middle points of the sides $BC$ and $AC$ and $G_1$ and $G_2$ the centres of gravity of the triangles $OAC$ and $OBC$ respectively.

In the $\triangle ODE$, $OG_1 = \frac{2}{3}OE$ and $OG_2 = \frac{2}{3}OD$, hence $G_1G_2$ is parallel to $ED$. But $ED$ is parallel to $AB$, hence $G_1G_2$ is parallel to $AB$.

The depths of the centres of gravity of the triangles $OAC$ and $OBC$ being the same, the total pressures on them will be proportional to their areas.

Now the area of $\triangle OCA = \frac{1}{2}OA.OC.\sin AOC$

$$= \tfrac{1}{2}R^2 \sin 2B,$$

and the area of $\triangle OBC = \frac{1}{2}OB.OC.\sin BOC$

$$= \tfrac{1}{2}R^2 \sin 2A.$$

Hence, the pressure on $\triangle OCA$ : the pressure on $\triangle OBC$

$$= \sin 2B : \sin 2A.$$

(iii) *A cubical vessel is filled with two liquids of densities* $\varrho$ *and* $\varrho'$, *the volume of each being the same. Find the pressure on the base and on one of the sides of the vessel.*

Suppose $a$ is the length of a side of the vessel.

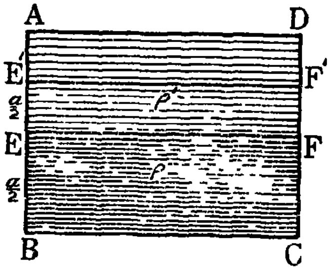
The pressure at a point on the base

$$= \tfrac{1}{2}a\varrho'g + \tfrac{1}{2}a\varrho g.$$

Hence the pressure on the base

$$= \tfrac{1}{2}a.a^2(\varrho' + \varrho)g$$
$$= \tfrac{1}{2}a^3(\varrho' + \varrho)g.$$

The pressure on one of the vertical sides, say $ABCD$, may be obtained by one of the two methods given below.




**First Method**

Imagine the liquid of density $\varrho'$ in contact with $AEFD$ to be replaced by a portion $E'EFF'$ of density $\varrho$ which would cause the same pressure at the level $EF$ that the given liquid of density $\varrho'$ does, so that

$$EE' \times \varrho g = \tfrac{1}{2}a.\ \varrho' g,$$

or

$$EE' = \frac{a}{2} \cdot \frac{\varrho'}{\varrho}.$$

Now the thrust on the portion $EBCF$

$$= \text{Area} \times \text{depth of C. G. of } EBCF \text{ below } E'F'$$
$$\times \text{ wt. of unit volume of the liquid of density } \rho$$
$$= \frac{a^2}{2} \times \left(\frac{a}{4} + \frac{a}{2} \cdot \frac{\rho'}{\rho}\right) \times \rho g$$
$$= \frac{a^3}{8}(\rho + 2\rho')g. \quad \ldots\ldots \quad (1)$$

The pressure on the portion $AEFD$

$$= a^2/2 \times a/4 \times \rho' g = 1/8 \times a^3 \rho' g \quad \ldots \quad (2)$$

Adding (1) and (2), we get the total thrust on $ABCD$

$$= (a^3/8) \times (3\rho' + \rho)g.$$

**Second Method**

The thrust on the side will remain the same even if we consider that one liquid of density $\rho'$ is in contact with the whole side and another liquid of density $(\rho - \rho')$ in contact with the lower half only of the side.

The thrust due to the first liquid

$$= a^2 \,.\, a/2 \,.\, \rho' g = a^3/2 .\, \rho' g.$$

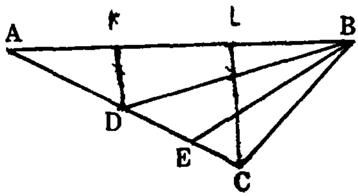
The thrust due to the second liquid

$$= a^2/2 \cdot a/4 .\, (\rho - \rho')g = 1/8 .\, a^3(\rho - \rho')g.$$

Adding, we get the thrust on $ABCD$

$$= a^3/8\ (4\rho' + \rho - \rho')g$$
$$= a^3/8 .\, (3\rho' + \rho)g.$$

(iv) *The side AB of a triangle ABC is in the surface of a fluid and points D, E are taken in AC, such that the pressures on the triangles BAD, BDE, BEC are equal; find the ratio AD: DE: EC.* [M. T.]




The depths of $D$, $E$ and $C$ are $AD \sin A$, $AE \sin A$ and $AC \sin A$ respectively.

If $w$ represents the weight of unit volume of the liquid, we have the pressure on the $\triangle ABD$

$$= \tfrac{1}{2}.AB.AD \sin A \times \tfrac{1}{3}.AD \sin A \times w$$
$$= K.AD^2.w, \quad \ldots\ldots \quad (1)$$

where $\quad K = \tfrac{1}{6}AB \sin^2 A.$

Similarly, the pressure on the $\triangle ABE$

$$= K.AE^2 w = 2K.AD^2.w, \quad \ldots\ldots \quad (2)$$

and the pressure on the $\triangle ABC$

$$= K.AC^2.w = 3K.AD^2.w, \qquad \ldots\ldots \quad (3)$$

since the pressures on the $\triangle^s$ $ABD$, $BDE$, $BEC$ are equal.

From (1), (2) and (3) we get

$$AD^2 = \frac{AE^2}{2} = \frac{AC^2}{3},$$

or

$$AD = \frac{AE}{\sqrt{2}} = \frac{AC}{\sqrt{3}},$$

or

$$\frac{AD}{1} = \frac{DE}{\sqrt{2}-1} = \frac{EC}{\sqrt{3}-\sqrt{2}}.$$

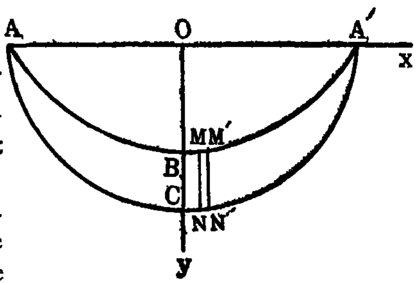
(v) *An ellipse is placed with its major axis on the surface of water and its plane vertical; a circle—the auxiliary circle—is described on the major axis as diameter. Find the thrust of water on the portion of the area enclosed between the auxiliary circle and the ellipse.*

Let the equations of the ellipse and the circle be respectively

$$x^2/a^2 + y^2/b^2 = 1,$$

and

$$x^2 + y^2 - a^2.$$




The depths of the C.G. of the semi-circle and the semi-ellipse being $4a/3\pi$ and $4b/3\pi$ respectively, we have

the required thrust on the area
= the thrust on the semi-circle
— the thrust on the semi-ellipse

$$= w.\frac{\pi a^2}{2}.\frac{4a}{3\pi} - w.\frac{\pi ab}{2}.\frac{4b}{3\pi}$$

$$= \tfrac{2}{3}wa(a^2 - b^2).$$

**Alternative Method**

The example can be done also in a different way by using the method of *Integral Calculus*. In this method the knowledge of the positions of the C.G. of a semi-circle and semi-ellipse will not be required.

Divide the enclosed area into thin elementary vertical strips like $MNN'M'$. Let the distance of $MN$ from $OY$ be $x$ and the thickness of the strip $MNN'M'$ be $dx$. Let the distance of $M$ from $AOA'$ be denoted by $y_m$.

The thrust on the strip

$= w \times$ area of the strip $\times$ depth of its C.G. below $AOA'$

$= w \times (y_n - y_m)\, dx \times \frac{1}{2}(y_n + y_m)$

$= \frac{1}{2}w(y_n^2 - y_m^2)\, dx.$

Since $N$ is on the auxiliary circle and $M$ on the ellipse, we have

$$y_n^2 = a^2 - x^2, \text{ and } y_m^2 = b^2(1 - x^2/a^2).$$

$$\therefore \quad y_n^2 - y_m^2 = (a^2 - b^2)(1 - x^2/a^2).$$

Hence, the total thrust for the enclosed area

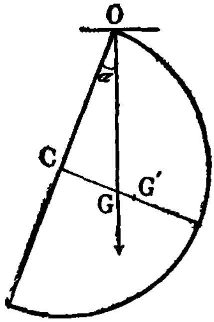
$$= \Sigma\, \tfrac{1}{2}w\,(a^2 - b^2)(1 - x^2/a^2)\, dx$$

$$= \int_{-a}^{a} \tfrac{1}{2}w\,(a^2 - b^2)(1 - x^2/a^2)\, dx$$

$$= \tfrac{2}{3}wa\,(a^2 - b^2).$$

(vi) *A hollow weightless hemisphere, filled with liquid, is suspended freely from a point in the rim of its base; prove that the whole pressure on the curved surface and the base are in the ratio* 19 : 8.

Let $a$ be the radius of the hemisphere and $O$ the point of the rim from which it is suspended. $G$ being the C. G. of the hemisphere, $CG = 3a/8$, and $OG$ must be vertical. If $\alpha$ be the inclination of the base to the vertical, then




$$\tan \alpha = \tfrac{3}{8}. \quad \ldots \quad (1)$$

The whole pressure on the base

$$= w\,\pi a^2 \,.\, a\cos\alpha. \quad \ldots \quad (2)$$

If $G'$ be the C.G. of the curved surface of the hemisphere, then $CG' = a/2$.

The depth of $G'$ below O

$$= a\cos\alpha + \tfrac{1}{2}a\sin\alpha.$$

Hence, the whole pressure on the curved surface

$$= w\,.\,(2\pi a^2)\,.\,(a\cos\alpha + \tfrac{1}{2}a\sin\alpha). \quad . \quad (3)$$

Comparing (2) and (3), we get the required ratio as

$$2(\cos\alpha + \tfrac{1}{2}\sin\alpha) : \cos\alpha,$$

or $\quad 2(1 + \frac{1}{2}\tan\alpha) : 1,$

or $\quad 2(1 + \frac{3}{16}) : 1 \quad$ from (1),

or $\quad 19 : 8.$

## Examples III

1. Find the thrust on a rectangular board, 3ft. by 2ft., immersed vertically in water with the smaller top edge horizontal and at a depth of 24·5 feet below the surface of water, the atmospheric pressure being equal to a column of 34 ft. of water.

2. Find the thrust on a square board whose side is 1 foot, when sunk in water to a depth of 20 feet, the board being horizontal and the atmospheric pressure at the surface being equivalent to a height of 30 inches of mercury. (Take the sp. gr. of mercury as 13·59)

3. A triangular area of 100 sq. feet has its vertices at depths of 5, 10 and 18 feet below the surface of the water. Find the resultant thrust on the area, the atmospheric pressure being 15 lbs. wt. per sq. inch.

4. Determine the total thrust on one side of a rectangular vertical dock-gate 50 feet wide, immersed in salt water to a depth of 25 feet, having given that one cubic foot of salt water weighs 1026 ozs.

5. The base of a rectangular tank, the upper edges being 2 ft. and 5 ft. in length, is inclined so that when the tank is full, it is 4 ft. deep at one edge and 2 feet at the opposite edge. Find the resultant pressure on the base.

6. In the vertical side of a water tank there is a square plate whose upper edge is horizontal and 8 ft. below the surface of the water. The depth of the plate is one foot; find the resultant pressure on the plate, taking the weight of 1 cu. ft. of water to be 62·5 lbs. [*Calcutta*, 1910]

7. A square plate, whose edge is 8 inches, is immersed in water, its upper edge being horizontal and at a depth of 12 inches below the surface of the water. Find the thrust of the water on the surface of the plate when it is inclined at 45° to the horizon; the mass of a cu. ft. of water being 64 lbs. [*Calcutta*, 1937]

8. Find the total thrust of water on a semi-circular area of radius 6 inches immersed in water with its diameter in the surface and the plane inclined to the vertical at an angle of 60°.

9. An ellipse is placed with its minor axis on the surface of water and its plane vertical. A circle is described on the minor axis as diameter. Find the total pressure of water on the portion of the area enclosed between the ellipse and the circle.

10. Taking a cubic foot of water as 1,000 ozs., find the pressure of the water arising from its weight on a side of a cistern 7 ft. wide and 8 ft. deep, if the cistern is filled with water.

What horizontal line would divide the side into two parts, so that the total pressure on each would be the same ?

11. The sp. gr. of sea-water olive oil and alchohol are 1·027, 0·915 and 0·795 respectively, the oil and alchohol have depths 1″ and 2″ above the water. Find the pressure on 3 sq. inches of a plane surface which is immersed horizontally at a depth of 5 inches below the upper surface of the oil, the weight of a cubic foot of distilled water being 1,000 ozs.

12. A triangle is immersed in a liquid with one vertex in the surface and the opposite side horizontal. Neglecting the atmospheric pressure, find the ratio in which the median through the vertex will be divided by a horizontal line which divides the triangle into two parts on which the total pressures are equal. [*Allahabad*, 1922]

13. A square is placed in a liquid with one side in the surface. Show how to draw a horizontal line in the square dividing it into two portions, the thrusts on which are the same. [*Calcutta*, 1938]

14. A square lamina $ABCD$, which is immersed in water, has the side $AB$ in the surface. Draw a line $BE$ to a point $E$ in $CD$, such that the pressures on the two portions into which it divides the lamina, may be equal. [*Agra*, 1931, 1937]

15. $ABCD$ is a rectangle immersed in a homogeneous liquid with $AB$ in the free surface and $AD$ vertical Show how to draw a straight line through $A$, dividing the rectangular area into two portions equally pressed. [*Nagpur*, 1943]

16. The side $AB$ of a triangle $ABC$ is in the surface of a fluid and a point $D$ is taken in $AC$, such that the pressures on the triangles $BAD$ and $BDC$ are equal, find the ratio $AD : DC$.

17. The sides of a cistern are vertical Its base is a horizontal regular hexagon each side of which is $\sqrt{3}$ ft. long. Find the depth if, when it is full of water, the thrust on each of its sides is the same as on its base. [*Agra*, 1936]

• 18. A triangle $ABC$ is immersed vertically in a liquid with the vertex $C$ in the surface, and the sides $AC$, $BC$ equally inclined to the surface, show that the vertical through $C$ divides the triangle into two others, the fluid pressures upon which are as

$$b^3 + 3ab^2 : a^3 + 3a^2b.$$

19. A hollow cone, whose axis is vertical and base downwards, is filled with equal volumes of two liquids whose densities are in the ratio 3 : 1; show that the thrust on the base is $(3-4^{1/3})$ times as much as it is when the vessel is filled with the lighter fluid.

[*Lucknow*, 1930]

20. The lighter of the two liquids of density $\rho$ rests on the heavier of density $\sigma$, to a depth of '$a$' inches. A square of side $b$ is immersed in a vertical position with one side in the surface of the upper liquid; if the thrusts on the two portions of the square in contact with the two liquids be equal, prove that

$$\rho a\,(3a - 2b) = \sigma(b - a)^2. \qquad \text{[\textit{Allahabad}, 1927]}$$

21. The lighter of two liquids, whose sp. gravities are as 2 : 3, rests on the heavier, to a depth of 4″. A square is immersed in a vertical position with one side in the upper surface; determine the side of the square in order that the thrusts on the portions in the two liquids may be equal. [*M.T.*]

22. Into a vessel containing a liquid of sp. gr. $\rho$ is poured water to a depth $a$. If a rectangular area of height $b$ is immersed vertically, part in the water and part in the lower liquid, find the length of the area in this liquid when the fluid pressures on the two portions are equal.

23. A rectangular area is immersed in a heavy liquid with two sides horizontal, and is divided by horizontal lines into strips on which the total thrusts are equal. Prove that, if $a$, $b$, $c$ are the breadths of three consecutive strips,

$$a(a + b)(b - c) = c(b + c)(a - b) \qquad \text{[\textit{Benares}, 1941]}$$

24. A parallelogram is immersed in a homogeneous liquid with one side in the surface; show how to draw horizontal lines dividing it into $n$ portions the thrusts on which are equal.

25. A semi-circle is immersed vertically in a liquid with the diameter in the surface; show how to divide it into n sectors, such that the thrust on each is the same. [*M.T.*]

26. A cube is filled with a liquid and held with a diagonal vertical, find the pressures on one of the lower and one of the upper faces.

27. Two dock-gates close a channel 12ft. wide, the depth of water on one side of the gate is 3ft. and on the other 15ft. Find in tons weight the force that must act on either gate to prevent them from opening.

28. The inclinations of the axis of a submerged solid cylinder to the vertical in two different positions are complementary to each other. If $P$ and $P'$ be the difference between the pressures on the two ends in the two cases, prove that the weight of the displaced fluid is equal to $(P^2 + P'^2)^{1/2}$.

29. A cylindrical tumbler, half filled with a liquid of density $\rho$, is filled up with a liquid of density $\rho'$ which does not mix with the former one. Show that the pressure on the base of the tumbler is to the whole pressure on its curved surface as $2r(\rho + \rho')$ to $h(\rho + 3\rho')$, where $h$ is the height and $r$ the radius of the base of the tumbler. [*Allahabad*, 1943]

30 A cylindrical vessel on a horizontal circular base of radius $a$, is filled with a liquid of density $w$ to a height $h$. If now a sphere of radius $c$ and density greater than $w$ is suspended by a thread so that it is completely immersed, find the increase of pressure on the base of the vessel, and show that the increase of the whole pressure on the curved surface is

$$(8\pi/3a)\times\ wc^3(h + 2c^3/3a^2).$$

31. A closed hollow cone is just filled with liquid, and is placed with its vertex upwards and axis vertical, divide its curved surface by a horizontal plane into two parts on which the whole pressures are equal.

32. A hollow weightless cylinder filled with water is suspended freely from a point on the rim of its plane end. If the height of the cylinder be twice the radius of the plane circular ends, show that the thrusts on the ends are in the ratio of 3 : 1.

33. A hollow weightless cone of semi-vertical angle $\alpha$ and of height $h$, is filled with liquid and freely hung from a point on the rim of the base; show that the thrust of the water on the base is

$$\frac{4\pi h^3 \tan^3\alpha \sin\alpha \,.\, w}{\sqrt{1 + 15\sin^2\alpha}}.$$

34. The centre board of a yacht is in the form of a trapezoid in which the two parallel sides are 3 and 5 ft. respectively, and the side perpendicular to these two is 4 feet in length. Assuming that the last named side is parallel to the surface of the water at a depth of 1 foot and that the parallel sides are vertical, find the total pressure on the board.

35. A parallelogram $ABCD$ is immersed in a homogenous fluid of density $\rho$, open to the atmospheric pressure $\Pi$, with the side $AB$ in the surface. $E$ is a point in $AB$ such that $AE = \frac{2}{3}AB$. A straight line joins $E$ to a point $F$ in $CD$. Show that, if the thrusts on $AEFD$ and $EBCF$ are equal, $DF$ is given by

$$6\,DF(3\Pi + 2g\rho b) = AB(6\Pi + 5g\rho b),$$

where $b$ is the depth of $AD$. [M.T.]

CHAPTER III

# RESULTANT THRUST ON CURVED SURFACES

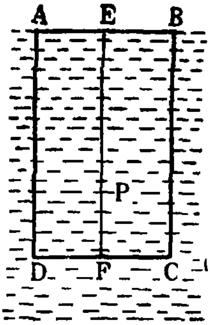
3·1. **Centre of Pressure.** We have seen that when a plane surface is in contact with a fluid, the latter exerts a pressure which is everywhere normal to the plane surface. The magnitude of the pressures on the various elements of the area depends upon the depths of these elements and hence will generally differ; but the direction of the pressure being always perpendicular to the plane area, these pressures will form a system of parallel forces which can be compounded into a single force acting at some definite point of the plane of the area. This single force gives the **Resultant Thrust** or **Resultant Pressure** of the fluid upon the *plane area* as already explained, and the point where its line of action meets the plane is called the **Centre of Pressure** of the plane area.

**Definition.** *The* **Centre of Pressure** *of a plane area in contact with a fluid is the point of the area at which the Resultant Thrust of the fluid on one side of the area acts.*

The determination of the positions of the centres of pressure for areas of different shapes will be discussed in the next chapter. But below we give the position of the centre of pressure (written briefly as C. P.) for a few simple standard cases leaving the proofs for the next chapter which is devoted wholly to the subject of centre of pressure.

3·11. For the sake of simplicity we are stating for the present the positions of the centres of pressure *neglecting the atmospheric pressure.*

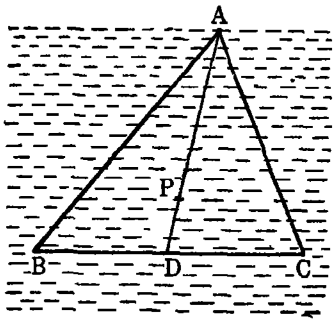
(i) **C. P. for a Rectangle.** Let a rectangular area




$ABCD$ be immersed in a homogeneous liquid with the side $AB$ in the surface.

*If E and F be the middle points of $AB$ and $DC$ respectively, then the centre of pressure of the rectangle $ABCD$ will be at the point P on $EF$, where $EP = \frac{2}{3}\, EF$.*

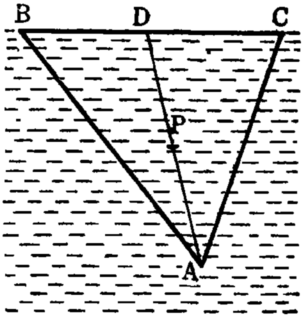
(ii) **C. P. for a Triangle with its vertex in the**




**surface and base horizontal.** Let a triangle $ABC$ be immersed in a liquid with the vertex $A$ in the surface and the base $BC$ horizontal.

*If D be the middle point of BC, then the C. P. of the triangle $ABC$ is the point P on $AD$, such that $AP = \frac{3}{4}\, AD$.*

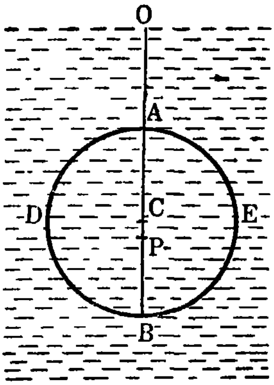
(iii) **C. P. of a Triangle with its base in the surface.**




Let a triangle $ABC$ be immersed in a liquid with its base $BC$ in the surface of the liquid.

*If $D$ be the middle point of $BC$, then the centre of pressure $P$ of the triangle $ABC$ bisects $AD$.*

(iv) **C. P. of a Circle.** Let a circular area $ADBE$




with radius $a$ be totally immersed in a homogeneous liquid with its plane vertical and its centre $C$ at a depth $h$, so that $OC = h$ and $AC = CB = a$, where $ACB$ is the vertical diameter.

*The centre of pressure $P$ of the circular area lies on $CB$, such that* $CP = \dfrac{a^2}{4h}$, or $OP = h + \dfrac{a^2}{4h}$.

**3·2. Thrust on a Curved Surface.** When a *curved* surface is in contact with a fluid, then, unlike the case of a plane surface, the thrusts on the various elements of the curved surface are neither in the same direction, nor generally in the same plane. Consequently this system of fluid pressures constituting the **Resultant Thrust,** can be reduced in general to a *force together with a couple,* and not to a *single* force. It can, however, be shown that this totality of pressures is equivalent to certain forces in the vertical and horizontal directions which can be determined as is explained below.

Supposing the surface divided into an indefinitely large number of small portions, let the fluid thrust on each element be firstly resolved into vertical and horizontal components. Now the vertical components at all the elements of the surface being parallel, form a system of parallel forces which can be compounded into a single resultant vertical force, say $Z$. This resultant is called the *Resultant Vertical Thrust* of the fluid upon the surface.

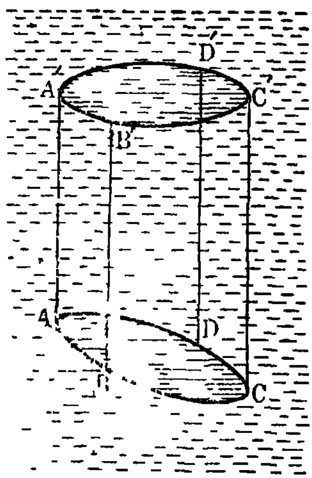
Next let us consider the horizontal components. Let $Ox$ and $Oy$ be two conveniently chosen horizontal directions at right angles to one another. Resolve parallel to $Ox$ and $Oy$ the horizontal components of fluid thrusts on all the elements of the surface. Then all the resolved components parallel to $Ox$ will form a system of parallel forces which can be combined into a single resultant $X$ acting parallel to $Ox$. Similarly the resolved components parallel to $Oy$ can be compounded into a single resultant $Y$ acting parallel to $Oy$. These resultants $X$ and $Y$ which can be thus determined in magnitude and direction, are called the *Resultant Horizontal Thrusts* of the fluid on the surface parallel to the assigned directions $Ox$ and $Oy$ respectively.

The fluid pressures on all the elements of the curved surface are thus equivalent to these three forces $X$, $Y$ and

$Z$ which together constitute the resultant of the fluid thrusts on the given surface. If in any particular case these three forces $X$, $Y$ and $Z$ become concurrent, then they can be combined into a single resultant force which may be called the **Resultant Thrust** of the fluid on the surface. If they are not concurrent, as is generally the case, then, as explained above, the Resultant Thrust is given by the forces $X$, $Y$ and $Z$.

We proceed now to show how these vertical and horizontal components can be determined.

3·3. **Resultant Vertical Thrust**. Let a surface be in contact with a liquid above it, and let the portion of the surface on which the vertical thrust is to be calculated be bounded by a curve $ABCD$. From every point of this boundary $ABCD$ conceive vertical lines to be drawn to meet in the curve $A'B'C'D'$ the surface of zero pressure (which is the free surface of the liquid or its effective surface, according as the atmospheric pressure is or is not neglected). These vertical lines together with the surface $ABCD$ and its projection $A'B'C'D'$ form a cylinder of liquid which may be called the 'superincumbent liquid.'



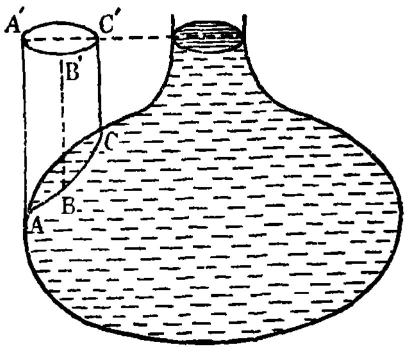

Considering the equilibrium of this cylinder of superincumbent liquid, we note that the only vertical forces on it are its weight acting downwards through its centre of gravity and the vertical component of the reaction of the surface $ABCD$ upon it acting upwards; these two must,

therefore, balance each other. But since the reaction of the surface is equal and opposite to the thrust of the liquid upon the surface, we conclude that the vertical component of the thrust of the liquid upon the surface *ABCD* is equal to the weight of the superincumbent liquid.

Hence, the rule for finding the resultant vertical thrust may be stated as follows :—

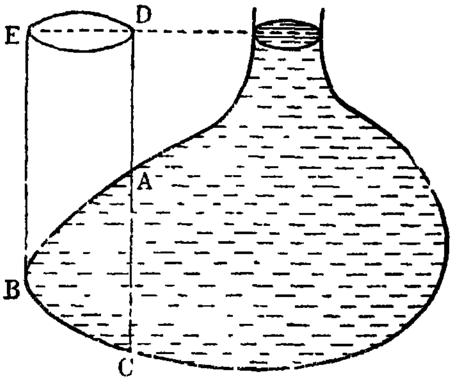
*The* **Resultant Vertical Thrust** *on any surface in contact with a liquid is equal to the weight of the superincumbent liquid and acts through the centre of gravity of this superincumbent liquid.*

**3·31. Liquid pressing upwards.** If the liquid, instead of being above the surface, be *below* it at every point as is the case for the surface *ABC* in the adjoining figure, then the liquid presses the surface *upwards*. Make the same construction for the superincumbent liquid by drawing verticals from every point of the boundary of *ABC* to meet the plane of the surface of zero pressure in the curve *A′B′C′*. Then the pressure at each point being due to its depth below the surface of zero pressure, it is clear that the vertical component of the *upward* thrust of the liquid on *ABC* is equal to the weight of the liquid which would fill the cylinder *ABCA′B′C′*.




Thus the rule arrived at in 3·3 for calculating the Resultant Vertical Thrust holds true always; if the liquid be above the surface the thrust acts *downwards* while, if the liquid be below the surface, it acts *upwards*.

**3·32. Surface being pressed partly upwards partly downwards.** Let us consider the vertical thrust on the portion $ABC$ of the surface in the given figure. Liquid being below the part $AB$, the resultant vertical thrust on it is *upwards* and equal to the weight of the liquid that would fill the space $ABED$.




Liquid is above the part $BC$, and consequently the resultant vertical thrust on it is *downwards* and equal to the weight of the liquid which would occupy the space $BCDE$.

The resultant vertical thrust on the surface $ABC$, being the difference of these, is thus equal to the weight of the liquid contained in $ABC$ and acts *downwards* through its centre of gravity

Even if the surface be of a complicated shape, by repeated use of the above method, the resultant vertical thrust on the surface can be easily obtained in magnitude and direction.

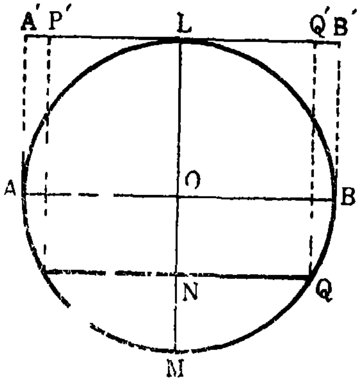
*Note.* It can be easily seen that the foregoing conclusions regarding the determination of the resultant vertical thrust will continue to hold true even if the liquid consists of layers of different liquids which do not mix.

**3·4. Illustrative Example.** *A hollow sphere of radius a is just filled with water; find the resultant vertical thrusts on the two portions into which the surface is divided by a horizontal plane at depth c below the centre.*

[*Allahabad*, 1940]

Since the sphere is just filled, the free surface of the liquid is the horizontal plane passing through $L$ which is the highest point of the sphere.

Let $PQ$ be the plane of division, so that $ON=c$. Let $AOB$ be the horizontal diametral plane. Draw verticals from the points of the circumference of the circular sections $AOB$ and $PNQ$ to meet the horizontal plane through the highest point $L$ of the sphere.




The thrust on the lower portion $PMQ$ is *downwards* and in magnitude

= wt. of the superincumbent liquid $P'PMQQ'$

= wt. of the cylinder $P'PQQ'$ of liquid

+ wt. of segment $PMQ$ of liquid

$$= \pi(a^2 - c^2)(a + c)w + \tfrac{1}{3}\pi(a - c)(3a^2 - a^2 - ac - c^2)w$$

$$= \tfrac{1}{3}\pi(a - c)\{3a^2 + 6ac + 3c^2 + 2a^2 - ac - c^2\}w$$

$$= \tfrac{1}{3}\pi(a - c)(5a^2 + 5ac + 2c^2)w. \quad . \quad . \quad . \quad . \quad (1)$$

The thrust on the upper portion $PLQ$ of the sphere is comprised of the *upward* pressure on $ALB$ and the *downward* pressure on the zone $APQB$.

The upward thrust on $ALB$

= wt. of the superincumbent liquid $A'ALBB'$

= wt. of cylinder $A'ABB'$ — wt. of hemisphere $ALB$

$$= \pi a^2 \,.\, a \,.\, w - \tfrac{2}{3}\pi a^3 w = \tfrac{1}{3}\pi a^3 w. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The downward thrust on the zone $APQB$

= wt. of superincumbent liquid

= wt. of the cylinder $AA'B'B$ + wt. of the zone $APQB$ — wt. of the cylinder $PP'Q'Q$

$$= \pi a^3 w + \tfrac{1}{3}\pi c(3a^2 - c^2)w - \pi(a^2 - c^2)(a + c)w$$

$$= \tfrac{1}{3}\pi(3a^3 + 3a^2c - c^3 - 3a^3 - 3a^2c + 3ac^2 + 3c^3)w$$

$$= \tfrac{1}{3}\pi(2c^3 + 3ac^2)w. \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Hence, the resultant vertical thrust on the upper portion being the difference of (2) and (3)

$$= \tfrac{1}{3}\pi w\{a^3 \sim (3ac^2 + 2c^3)\}.$$

It is upwards or downwards according as (2) is greater or less than (3), which will depend upon the relative values of $a$ and $c$.

## Examples IV

1. A conical vessel 10 inches high on a flat circular base of 5 inches radius, is filled with water. Calculate the vertical thrust on the base when the vertex is upwards

2. A conical vessel filled with water, stands with its plane base on a horizontal table. Prove that the thrust of the liquid on the base of the cone is three times the weight of the contained water. How do you explain the result ? [*Agra*, 1933]

3. A conical wineglass is filled with water and placed in an inverted position upon a table, how that the resultant thrust of the water on the glass is two-thirds that on the table. [*M.T.*]

4. A hollow cone filled with water and closed, is held with its axis horizontal; find the resultant vertical pressure on the upper half of its curved surface.

Find it on the lower half as well. [*Calcutta*, 1915]

5. A bucket in the form of a frustum of cone is filled with water. If the top and the bottom ends be of radii $a$ and $b$ ($b < a$) and the height $h$, find the resultant vertical thrust on the curved surface

Calculate the value of the pressure if $a = 8''$, $b = 5''$, and $h = 14''$; 1 cu. ft of water weighs 1000 ozs

6. Find the resultant vertical fluid thrust on the lower half of the curved surface of a cylindrical pipe of length $h$ and radius $r$ when (i) the pipe is full, (ii) when the height of the water above the bottom is $d$ ($d > r$)

7. A vessel in the shape of a hollow hemisphere surmounted by a cone is held with the axis vertical and vertex uppermost. If it be filled with a liquid so as to submerge half the axis of the cone in the liquid, and the height of the cone be double the radius of its base, show that the resultant upward thrust of the liquid on the vessel is 15/8 times the weight of the liquid that the hemisphere can hold.

8. The shape of the interior of a vessel is a double cone, the ends being open and the two portions being connected by a minute aperture at the common vertex. It is placed with one circular rim fitting close upon a horizontal plane and is filled with water. Find the resultant vertical thrust upon it, and prove that if it be zero, the ratio of the axes of the two portions is 1 : 2.

9. A conical cup whose weight is 5/8th of the weight of water which would just fill it, is placed vertex upwards on a smooth table and water is gradually poured in through a hole made in the top.

Show that the cup will be on the point of rising from the table when the water reaches half the height of the cup. [*Agra*, 1939]

10. A hollow cone is placed with its vertex upwards on a horizontal table and liquid is poured in through a small hole in the vertex; if the cone begins to rise when the weight of the liquid poured in is equal to its own weight, prove that its weight is to the weight of the liquid required to fill the cone, as $9 - 3\sqrt{3} : 4$. [*Patna*, 1941]

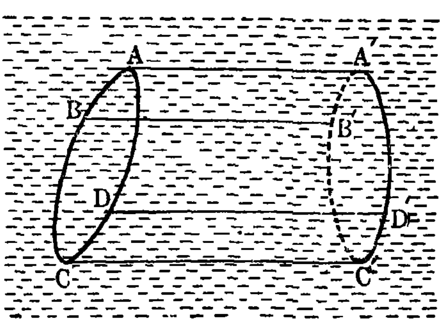
11. A pyramid with a square base and with sides which are equilateral triangles is placed on a horizontal plane and filled with a liquid through an aperture in the vertex; find the pressure on one of the sides.

If the pyramid has no base, find its least weight consistent with its not being raised.

12. A double-funnel is formed by joining two equal hollow cones at their vertices and stands on a horizontal plane with the common axis vertical; liquid is poured into the cone until its surface bisects the axis of the upper cone. If the liquid be on the point of escaping between the lower cone and the table, prove that the weight of either cone is to that of the liquid it can hold as 27 : 16. [M.T.]

**3·5. Resultant Horizontal Thrust.** Let a surface *ABCD* be in contact with a liquid. It is required to find the resultant horizontal thrust on the surface in an assigned horizontal direction.

Through every point of the perimeter of the surface *ABCD* draw horizontal lines in the assigned direction to meet a vertical plane perpendicular to them in the closed curve *A′B′C′D′*. Then the plane curve *A′B′C′D′* will become the projection of the surface *ABCD* on this vertical plane.



Now consider the equilibrium of the mass of liquid enclosed between the surface $ABCD$, its projection $A'B'C'D'$ and the cylindrical surface generated by lines drawn in the assigned horizontal direction. Since the only horizontal forces acting on this mass of liquid in equilibrium in the assigned horizontal direction are the horizontal component in the assigned direction of the thrust on the surface $ABCD$ and the thrust on the plane end $A'B'C'D'$, these two forces must balance one another. Now the thrust on the plane surface $A'B'C'D'$ is the *whole pressure* on it passing through its *centre of pressure*. Hence the resultant horizontal thrust on the surface $ABCD$ in the assigned direction is equal in magnitude to the whole pressure on $A'B'C'D'$; its line of action passes through the centre of pressure of $A'B'C'D'$ *from* the liquid *towards* the surface $ABCD$.

Thus the rule for finding the resultant horizontal thrust may be formulated as follows:—

**The Resultant Horizontal Thrust on a given surface in contact with a liquid in an assigned horizontal direction is equal, in magnitude and line of action, to the whole pressure on the projection of the surface upon a vertical plane perpendicular to the assigned direction.**

**Note.** While establishing the above theorem, it has been assumed, as is the case with most of the ordinary surfaces, that each of the horizontal lines like $AA'$ cuts the surface in one point only. If the case be otherwise, then the given surface should be divided into two or more parts so that the above theorem may be applicable separately to each part. The horizontal thrust in the assigned direction for each of the parts must be found separately, which after being compounded, will give the resultant horizontal fluid thrust on the whole of the given surface in the assigned direction.

**3·51. Resultant Thrust.** We have explained in 3·2 how the resultant thrust on a curved surface can be determined in terms of $X$, $Y$, $Z$. We have shown now

how the vertical component $Z$ and the horizontal component, say $X$, in an assigned direction can be found. In exactly the same manner as explained in 3·5, the resultant horizontal thrust in a direction at right angle to the previously assigned direction can be obtained, which may be called the component $Y$. Thus the Resultant Thrust on any surface in contact with a liquid can be completely determined in terms of the resultant vertical and horizontal thrusts on the surface.

We shall see in the examples which follow that for certain simple, symmetrical bodies, these three forces can be compounded into a single one, and then the Resultant Thrust can be obtained as a single force.

**3·6. Resultant Thrust on a Solid—Principle of Archimedes.** *The important theorem, known as the* **Principle of Archimedes,** may be stated as :—

**The resultant fluid thrust on a solid body, wholly or partially immersed in a fluid at rest, is equal to the weight of the fluid displaced by the body, and acts vertically upwards through the centre of gravity of the fluid displaced.**

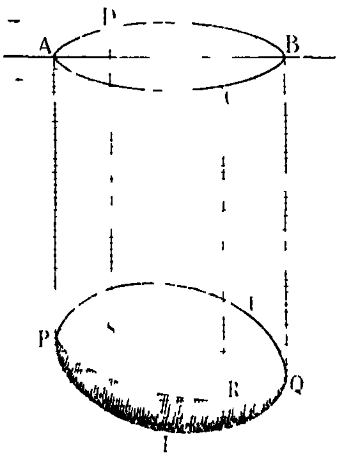
In order to prove the theorem it should be observed that the fluid thrust on a body does not depend upon the substance of which the body is made; it depends only on the shape of the body, its position and the fluid surrounding it. If instead of one particular body, any other body of exactly the same shape be placed in that very position in the fluid, the fluid thrust on this replaced body will be the same as before.

Now imagine the body to be removed and the space occupied by it to be filled up with extra fluid of the same kind as the surrounding fluid, the rest of the fluid remaining undisturbed. This mass of the extra fluid which may be called the 'displaced fluid' will form a continuous mass

with the surrounding fluid and would be in equilibrium under the action of its weight acting downwards and the thrusts of the surrounding fluid upon it. Hence the resultant thrust of the fluid upon any solid body that would fill the same space must be equal and opposite to the weight of the fluid displaced and must act upwards through the centre of gravity of the displaced fluid. This proves the theorem.

3·61. **Second Proof of Archimedes' Principle.** This theorem can also be proved by methods dealt with in 3·3 and 3·5.

Suppose a body *PTQU* is wholly immersed in a fluid. Suppose a vertical line going round the surface of the body touches it in the curve *PRQS* and meets the surface of zero pressure of the fluid in the curve *ACBD*. The whole surface of the body is then divided in two parts—*PTQRP* which is below the dividing curve and *PUQRP* which is above it. The surface *PTQRP* has fluid below it and the surface *PUQRP* above it.



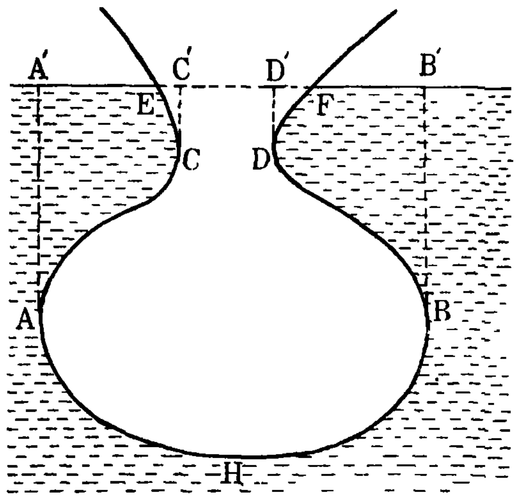

Now the resultant vertical thrust on the surface *PTQRP* is by 3·3 equal to the weight of the fluid that would occupy the space *PTQBA* and acts *upwards* through its C. G., while the resultant vertical thrust on the surface *PUQRP* is equal to the weight of the fluid filling the space *PUQBA* and acts *downwards* through its C. G. The *resultant vertical thrust* on the whole body being the resultant of these two upward and down-

ward thrusts, is equal to the weight of the fluid that would occupy the space $PTQU$, and acts *upwards* through the C. G. of this displaced fluid.

It can be easily seen that the *resultant horizontal thrust* on the surface of this body in *every* direction will be zero. For, following the method of 3·5 a vertical plane perpendicular to an assigned direction can be drawn, and the body divided into two parts so that the projections of the two parts into which the surface of the body is divided, on the plane will be exactly identical; and consequently, the horizontal thrusts on these two parts of the surface will be equal and opposite and act along the same line. Thus the *resultant horizontal thrust* will vanish.

Since the resultant horizontal thrust in every direction vanishes, the *resultant thrust* on the body is the resultant vertical thrust which, as shown above, is equal to the weight of the fluid displaced and acts upwards through its C.G.

Even if the body be only partially immersed, it can be




easily seen that the theorem holds true. Consider, for

instance, the partially immersed body of rather irregular shape as given in the figure.

Here the resultant vertical thrust on the body
= wt. of fluid $A'AHBB'$ acting upwards
− wt. of fluid $(ACC'A'+BDD'B')$ acting downwards
+ wt. of fluid $(ECC'+DFD')$ acting upwards
= wt. of fluid $ECAHBDF$ acting upwards
= wt. of the displaced fluid.

The resultant horizontal thrust being zero as shown above, the theorem is proved to be true even when the body is partially immersed.

Thus the theorem is completely proved.

**Note.** It must be noted that in case the fluid in which the body is immersed is not homogeneous, but consists of a number of layers of different fluids of varying densities which do not mix with one another, the mass of the displaced fluid should be taken to be of the same density at any point of it as that of the surrounding fluid at the same horizontal level.

**3·62. Definition. The resultant fluid thrust on a body wholly or partially immersed in a fluid is called the Force of Buoyancy, and the centre of gravity of the fluid displaced is called the Centre of Buoyancy of the body.**

**3·63.** The important theorem proved in **3·6** and **3·61** may be considered to be the first established theorem of Hydrostatics. This principle is said to have been discovered by Archimedes in his bath from observations on the buoyancy of his own body. The proof given in **3·6** is based on the method employed by Archimedes himself.

**3·64. Thrust on a vessel containing liquid.** Suppose there is some liquid contained in a vessel. This mass of liquid is in equilibrium under the action of its weight and the reaction of the vessel upon the liquid. But this reaction of the vessel upon the liquid being equal and

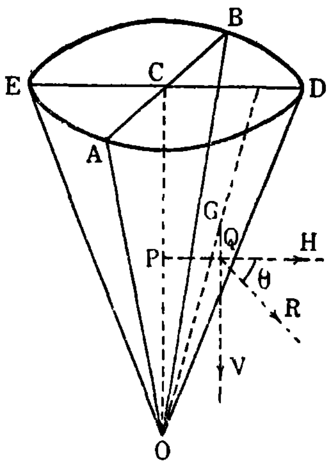
opposite to the thrust of the liquid upon the inner surface of the vessel, we conclude that the thrust of the liquid on the vessel is equal to the weight of the liquid contained and acts vertically downwards through the C. G. of the contained liquid.

Hence the rule is:

*If a liquid be contained in a vessel, then the resultant thrust of the liquid on the vessel is equal to the weight of the liquid contained in the vessel and acts vertically downwards through the centre of gravity of the liquid.*

3·7. **Illustrative Example.** *A hollow right circular cone filled with liquid is held with its axis vertical and vertex downwards. Find the magnitude and the line of action of the resultant fluid thrust on half the surface of the cone cut off by a vertical plane through the axis.*

In the figure, $OAB$ is the vertical plane of separation of the cone $ODE$, so that it is required to find the resultant thrust on the surface of the semi-cone $OADB$. Let $h$ be the height and $r$ the radius of the base of the cone.




Choosing $AB$ and its perpendicular $CD$ as the two horizontal directions along which components of horizontal thrusts may be taken, we note that the horizontal thrust in the direction $AB$ must be zero. For, if the surfaces $OAD$ and $OBD$ be projected on the vertical plane $OCD$, which is a plane of symmetry for the semi-conical surface, the projections will be identical; consequently the horizontal thrusts for the surfaces $OAD$ and $OBD$, being equal and opposite, will cancel one another. Therefore the horizontal thrusts will be parallel to $CD$ only.

Let $V$ and $H$ denote the resultant vertical and horizontal thrusts respectively.

Now $V$ = wt. of the superincumbent liquid
= wt. of the vol. $OADB$ of liquid
$= (h/2 \cdot 3)\, \pi r^2 w,$ . . (1)

where $w$ represents the weight of unit volume of the liquid.

Again, $H$ = whole pressure on the projection of the semi-conical surface $OADB$ on the vertical plane $OAB$
= whole pressure on the triangle $OAB$
$= w.rh.\, h/3 = rh^2w/3.$ . . . . . (2)

This $H$ will be acting through the centre of pressure $P$ of the triangle $OAB$, where $CP = h/2$; its line of action will be in the plane $OCD$ perpendicular to $OC$.

The vertical thrust $V$ acts downwards through the centre of gravity $G$ of the semi-cone which is in the plane $OCD$, its vertical line of action being parallel to $OC$ at a distance $r/\pi$ from it.

Both $V$ and $H$ being in the same vertical plane $OCD$, they intersect at a point, say $Q$, such that its distances from $OC$ and $CD$ are $r/\pi$ and $h/2$ respectively.

Now if R represents the resultant thrust which passes through $Q$, and $\theta$ its angle of inclination to the horizontal, we get

$$R = \sqrt{V^2 + H^2}$$
$$= \tfrac{1}{6}\, wrh \sqrt{\pi^2 r^2 + 4h^2},$$

and $$\theta = \tan^{-1}(V/H) = \tan^{-1}(\pi r/2h).$$

## Examples V

1. A right circular cone filled with liquid is held with its axis vertical. Prove that the horizontal thrust on half the curved surface cut off by a plane through the axis, when the vertex is upwards is twice that when the vertex is downwards.

2. A right circular cone is just immersed in a liquid with its axis horizontal. Find the resultant horizontal thrust (i) on half the cone cut off by a vertical plane through the axis, (ii) on lower half of the cone cut off by a horizontal plane through the axis.

3. A solid circular cylinder is divided in two equal parts by a plane through the axis. If it is held just immersed in a liquid with the axis horizontal and the plane section vertical, what will be the resultant horizontal thrust on the curved surface?

4. A hollow cylinder closed by a plane base is filled with liquid and held with its axis vertical; find the magnitude and the line of ac-

tion of the resultant thrust on half the cylinder cut off by a vertical plane through the axis.

5. A hollow right circular cylinder is filled with liquid and held with its axis horizontal; find the magnitude and the line of action of the resultant thrust on half the curved surface cut off by a vertical plane through the axis. [Agra, 1943]

6. A hemispherical bowl with its lowest point downwards and the plane base horizontal is filled with water. The water is poured into a cylindrical tumbler, the radius of whose base is equal to that of the hemisphere. Prove that the horizontal thrusts on half of their curved surfaces in which they may be divided by vertical planes through their axes, are in the ratio of 3 : 2.

7. A hemispherical bowl is filled with water; find the horizontal fluid pressure on one-half of the surface divided by a vertical diametral plane, and show that it is $1/\pi$ of the magnitude of the resultant fluid thrust on the whole surface. [*Lucknow*, 1941]

8. A cylindrical pipe of circular cross section is half full of water. If the pipe be imagined to be divided into two halves by a vertical plane along the middle, show that the water will tend to push them asunder horizontally with a force $W/\pi$, where $W$ is the weight of the water contained. Show that the resultant thrust of the water on either half of the pipe makes with the vertical an angle $\cot^{-1}(\pi/2)$.

9. A closed cylindrical vessel with hemispherical ends is filled with water, and placed with its axis horizontal. Find the resultant thrust on each of the ends and determine its line of action.

[*Allahabad*, 1926]

10. A hemisphere of radius $a$ is immersed in a liquid of density $\sigma$. The plane of the base is vertical and its centre at a depth $a\sqrt{5}$ below the surface. Show that the resultant force on the curved surface is $\frac{7}{3}\pi\sigma g a^3$ and that its direction makes with the horizontal an angle $\theta$, where $\tan\theta = 2/\sqrt{45}$. [*M. T.*]

11. A right circular cone is divided into two parts by a plane through its axis. One of these portions is just immersed vertex downwards in water. Find the resultant thrust on its curved surface, and show that it is inclined at an angle $\tan^{-1}(\frac{1}{2}\pi \tan\alpha)$ to the horizontal, where $\alpha$ is the semi-vertical angle of the cone.

[*Agra*, 1929; *Allahabad*, 1933].

12. The end of a horizontal pipe is closed by a sphere of the same radius $a$ as the internal section of the pipe. The sphere is

hinged at its highest point. If the pipe is just full of liquid of density $\varrho$, prove that the moment about the hinge of the liquid pressure on the sphere is $g\varrho\pi a^4$. [*M. T.*]

13. A spherical shell formed of two halves in contact along a vertical plane is filled with water; show that the resultant pressure on either half of the shell is $\sqrt{13}/4$ of the total weight of the liquid.

14. Two closely fitting hemispheres made of sheet metal of small uniform thickness are hinged together at a point on their rims, and are suspended from the hinge, the rims being greased so that they form a water-tight spherical shell; this shell is now filled with water through a small aperture near the hinge. Prove that the contact will not give way if the weight of the shell exceeds three times that of the water it contains. [*M. T.*]

15. A hollow right circular cone is divided into two parts by a plane through the axis, and the two parts are hinged together at the vertex, the edges being greased so as to be water-tight. The vessel is then hung up by the hinge and filled up with water through a small aperture near the hinge. If the water does not flow out, what must be the least value of the vertical angle of the cone?

16. A solid circular cone of uniform material and height $h$ and of vertical angle $2\alpha$, floats in water with its axis vertical and vertex downwards and a length $h'$ of the axis immersed. The cone is bisected by a vertical plane through the axis and the two parts are hinged together at the vertex. Show that the two parts will remain in contact, if $h' > h \sin^2 \alpha$.

[*Allahabad*, 1938; *Agra*, 1934, *M T.*, *Benares*, 1943]

**3·8. Thrust on a curved surface bounded by a plane curve.** When a curved surface in contact with a liquid is bounded by a plane curve, the resultant thrust on it may often be conveniently obtained in the following manner without using the method of 3·51.

If some liquid is enclosed by the given curved surface $S$ and a plane boundary $A$, then we know from 3·64 that the resultant thrust of the liquid on $S$ *together with* its thrust on $A$ must be equal to the weight of the liquid enclosed. Hence by calculating the weight of the liquid and the thrust on $A$, which is the *whole pressure* to be

obtained by the formula of 2·81, the resultant thrust on $S$ can be found.

If a solid body enclosed by a curved surface $S$ and a plane boundary $A$ be immersed in a liquid, then in this case the resultant thrust on $S$ together with the *whole pressure* on $A$ will be equal to the weight of the liquid displaced, by Archimedes' Principle, and the above method for finding the thrust on $S$ will be still applicable.

The method is illustrated by means of examples in the next article.

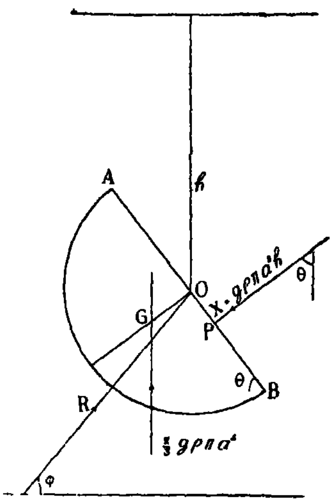
**3·9. Illustrative Examples.** (1) *A hemisphere, radius $a$, is entirely submerged in a liquid of density $\rho$ so that its diametral plane makes an angle $\theta$ with the horizontal and has its centre at a depth $h$. Prove that the resultant force on the curved surface is*

$$\pi a^2 g\rho(\tfrac{4}{9}a^2 + h^2 \pm \tfrac{4}{3}ah \cos\theta)^{1/2}.$$

[*I.C.S.*, 1935; *M.T.*]

*Find also the direction of the resultant thrust.*

By Archimedes' Principle, the resultant thrust on the whole surface of the body is equal to the weight of the liquid displaced, viz., $\frac{2}{3}g\rho\pi a^3$, and acts vertically upwards through the centre of gravity $G$ of the hemisphere.




But since the surface of a hemisphere is partly curved and partly plane, this thrust is the resultant of the following:—

(1) The whole pressure $X$ on the circular plane base acting normal to it at its centre of pressure $P$. By the formula of 2·81, we get

$$X = g\rho\pi a^2 h.$$

(2) The thrusts of the liquid acting on the elements of the curved surface. Since the curved surface is spherical and the direction of pressure is everywhere normal to the surface, the pressures on the various elements must pass through the centre O, and consequently the direction of the resultant thrust on the curved surafce must also pass through O. Let this resultant thrust be represented by R acting at an angle $\phi$ to the horizontal.

Resolving R and $X$ which together make up the total vertical thrust $\frac{2}{3}g\rho\pi a^3$, we get

$$\mathrm{R}\sin\phi - X\cos\theta = \tfrac{2}{3}g\rho\pi a^3, \qquad \dots\dots (1)$$

$$\mathrm{R}\cos\phi - X\sin\theta = 0. \qquad \dots\dots (2)$$

$\therefore$

$$\mathrm{R}\sin\phi = \tfrac{2}{3}g\rho\pi a^3 + X\cos\theta$$

$$= \tfrac{2}{3}g\rho\pi a^3 + g\rho\pi a^2 h\cos\theta;$$

and

$$\mathrm{R}\cos\phi = X\sin\theta = g\rho\pi a^2 h\sin\theta.$$

Hence

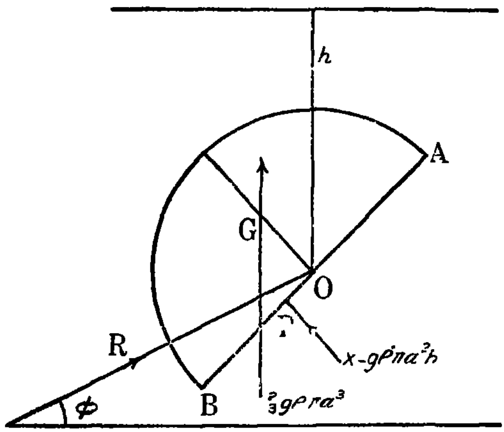
$$\mathrm{R} = g\rho\pi a^2\sqrt{(h\cos\theta + \tfrac{2}{3}a)^2 + (h\sin\theta)^2}$$

$$= g\rho\pi a^2\sqrt{\tfrac{4}{9}a^2 + h^2 + \tfrac{4}{3}ah\cos\theta};$$

and

$$\tan\phi = \frac{\mathrm{R}\sin\phi}{\mathrm{R}\cos\phi} = \frac{2a + 3h\cos\theta}{3h\sin\theta}.$$

If the plane base be as in the annexed figure, then the thrust $X$ acts upwards and the equations corresponding to (1) and (2) will be




$$\mathrm{R}\sin\phi + X\cos\theta = \tfrac{2}{3}g\rho\pi a^3,$$

$$\mathrm{R}\cos\phi - X\sin\theta = 0.$$

In that case

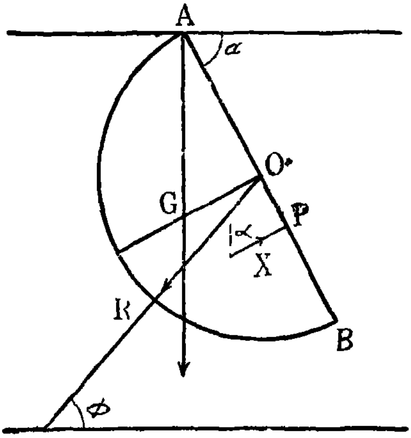
$$R = g\varrho\pi a^2\sqrt{\tfrac{4}{9}a^2 + b^2 - \tfrac{4}{3}ab\cos\theta},$$

and

$$\tan\phi = \frac{2a - 3b\cos\theta}{3b\sin\theta}.$$

(ii) *A hollow weightless hemisphere with a plane base is filled with water and hung up by means of a string, one end of which is attached to a point of the rim of its base; find the inclination to the horizontal of the resultant thrust on its curved surface.*

Let $a$ be the radius and $A$ the point of attachment of the hemisphere; then $AG$ will be vertical, where $G$ is the C. G. of the enclosed hemispherical mass of water, so that $OG = 3a/8$.




In this case the liquid being *inside* the hemisphere, the resultant thrust on the whole surface of the hemisphere is equal to the weight of the water contained, viz., $\tfrac{2}{3}\pi a^3 w$, where $w$ is the weight of unit volume of water, and it acts vertically downwards through $G$. This total thrust $\tfrac{2}{3}\pi a^3 w$ is the resultant of the thrust $X$ on the circular plane base and the thrust R on the curved surface of the hemisphere, which, as explained in the previous example, must be passing through the centre $O$. The directions of R and $X$ are as shown in the figure.

Let the inclination of R and of the plane base to the horizontal be $\phi$ and $\alpha$ respectively. Then

$$\tan\alpha = \cot\angle OAG = OA/OG = \tfrac{8}{3}. \quad \ldots\ldots (1)$$

Resolving R and $X$ vertically and horizontally, we get

$$R\sin\phi - X\cos\alpha = \tfrac{2}{3}\pi a^3 w,$$

$$R\cos\phi - X\sin\alpha = 0.$$

$$\tan\phi = \frac{\tfrac{2}{3}\pi a^3 w + X\cos\alpha}{X\sin\alpha}. \quad \ldots\ldots (2)$$

Now $X$ being the whole pressure on the circular base, we have by 2·81,

$$X = w.\pi a^2.a\sin\alpha = \pi a^3\sin\alpha.w.$$

Putting the above value of $X$ in (2), we get

$$\phi = \tan^{-1}\frac{\frac{2}{3}\pi a^3 w + \pi a^3 \sin\alpha\cos\alpha . w}{\pi a^3 \sin^2\alpha . w}$$

$$= \tan^{-1}\frac{2 + 3\sin\alpha\cos\alpha}{3\sin^2\alpha}.$$

From (1), we have

$$\frac{\sin\alpha}{8} = \frac{\cos\alpha}{3} = \frac{1}{\sqrt{73}}.$$

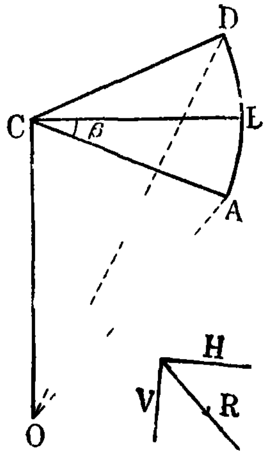
Hence, finally

$$\phi = \tan^{-1}\frac{2 + 3 \times \frac{8}{\sqrt{73}} \times \frac{3}{\sqrt{73}}}{3 \times \frac{64}{73}}$$

$$= \tan^{-1}\frac{2 + \frac{72}{73}}{3 \times \frac{64}{73}} = \tan^{-1}\frac{218}{3 \times 64} = \tan^{-1}\frac{109}{96}.$$

(iii) *A hollow right circular cone filled with water, is held with the axis vertical and vertex downwards. Find the resultant pressure on the portion of the surface contained between two vertical planes through the axis, and show that if the inclination of these planes to each other be $2\beta$ and the vertical angle of the cone $2\alpha$, the direction of this resultant pressure makes with the vertical an angle equal to*

$$\tan^{-1}\left(\frac{\sin\beta}{\beta\tan\alpha}\right).$$ [*Allahabad*, 1919]




Suppose $OABDC$ is the portion of the cone filled with water. Let $OBC$ be the vertical plane of symmetry which bisects the angle between the two vertical planes $OAC$ and $ODC$.

By symmetry the resultant thrust on the curved surface must lie in the plane $OBC$.

If $P$ be the thrust on each of the planes $OAC$ and $ODC$ then their components normal to the plane $OBC$ will cancel one another, and their resolved parts in the plane $OBC$ parallel to $BC$

$$= 2P\sin\beta$$

$$= 2\left(\frac{h.h\tan\alpha}{2} \times \frac{h}{3} \times w\right)\sin\beta$$

$$= \frac{h^3\tan\alpha\sin\beta}{3}w,$$

where $w$ is the weight of unit volume of water and $h$ the height of the cone.

The resultant thrust on the portion $OABDC$ of the cone is acting vertically downwards and

$$= \text{the wt. of the liquid contained}$$

$$= \tfrac{1}{3}h \,.\, \beta r^2 \times w = \frac{\beta h^3 \tan^2 \alpha}{3} w,$$

$r$ being the radius of the base of the cone.

This resultant thrust is produced by the resultant thrust $R$ on the *curved* surface $OABD$ and the thrusts on the two *plane* surfaces $OCA$ and $OCD$. If $H$ and $V$ denote the horizontal and vertical components of $R$, we have

$$H = 2P \sin \beta = \frac{h^3 \tan \alpha \sin \beta}{3} \,.\, w,$$

$$V = \frac{\beta h^3 \tan^2 \alpha}{3} w.$$

Hence

$$R = \sqrt{(H^2 + V^2)} = \frac{wh^3 \tan \alpha}{3} \sqrt{\sin^2 \beta + \beta^2 \tan^2 \alpha}\,.$$

Finally, the angle of inclination of the direction of the resultant thrust on the curved surface to the vertical

$$= \tan^{-1} (H/V)$$

$$= \tan^{-1} \left[\frac{h^3 \tan \alpha \sin \beta\, w}{3} \div \frac{\beta h^3 \tan^2 \alpha}{3} w\right]$$

$$= \tan^{-1} \left(\frac{\sin \beta}{\beta \tan \alpha}\right).$$

## Examples VI

1. Find the direction and magnitude of the resultant thrust on the curved surface of a hemisphere of radius 3″, placed with its base vertical and centre at a depth of 6″ below the free surface of a liquid of which one cubic inch weighs $w$ grammes. [*Agra*, 1938]

2. A solid hemisphere is immersed in a liquid with the highest point of its plane base in the surface, and the base is inclined at an angle $\tan^{-1} 2$ to the horizon; show that the resultant thrust on the curved surface is equal to twice the weight of the displaced liquid. [*Agra*, 1928]

3. A cylindrical vessel full of water is held with its axis inclined at an angle of $45°$ to the vertical. Find the magnitudes of the pressures on the ends and show that the resultant pressure on the curved surface will equal the difference between the pressures on the ends. [*Patna*, 1931]

4. A spherical shell is filled with liquid, find the magnitude and line of action of the resultant pressure on each of the hemispheres divided by any diametrical plane.

5. A cone floats with its axis horizontal in a liquid of density double its own; find the pressure on its base and prove that if $\theta$ be the inclination to the vertical of the resultant thrust on the curved surface, and $\alpha$ the semi-vertical angle of the cone, then

$$\tan\theta = (4/\pi)\tan\alpha.$$ [*Lucknow*, 1938]

6. A closed cylinder, whose base diameter is equal to its length, is full of water, and hangs freely from a point in its upper rim; prove that the vertical and horizontal components are each half the weight of the water. [*M.T.*]

7. A hollow cone without weight, closed and filled with water, is suspended from a point in the rim of its base; if $\phi$ be the angle which the direction of the resultant pressure makes with the vertical, then show that

$$\cot\phi = \frac{28\cot\alpha + \cot^3\alpha}{48},$$

$\alpha$ being the semi-vertical angle of the cone

8. A solid right circular cone of vertical angle $2\alpha$ is just immersed in water so that one generator is in the surface of the liquid; prove that the resultant pressure on the curved surface of the cone is to the weight of the fluid displaced by the cone, as $\sqrt{1 + 3\sin^2\alpha} : 1$, and that, it is inclined to the axis of the cone at an angle $\cot^{-1}(2\tan\alpha)$.

9. A cone whose vertical angle is $2\alpha$, has its lowest generator horizontal and is filled with liquid; prove that the resultant pressure on the curved surface is $\sqrt{1 + 15\sin^2\alpha}\ W$, where $W$ is the weight of the liquid.

10. A solid octant of a sphere is immersed with one plane face in the surface; prove that the resultant pressure on the curved surface is $(1 + 8/\pi^2)^{1/2}$ times the weight of water displaced by the octant.

11. A solid cone, whose vertical angle is $2\alpha$, is immersed in a liquid with its vertex in the surface and axis vertical. Prove that if

$\varrho$ be the whole pressure on the curved surface and base, and $\varrho'$ the resultant pressure, then

$$\varrho : \varrho' :: (2 + 3 \sin \alpha) : \sin \alpha. \qquad [\textit{Bombay}, 1936]$$

12. A solid sphere of density $\varrho$ is placed at the bottom of a vessel which is horizontal, and a liquid of density $\sigma\ (< \varrho)$ is poured in so as just to cover up the sphere. The sphere is then cut along the vertical diametral plane. Prove that two parts will not separate if $\varrho > 4\sigma$.

CHAPTER IV

# CENTRE OF PRESSURE

**4·1. The Problem.** When a plane area is in contact with a liquid, the pressures acting everywhere normal to the area form a system of parallel forces, and in 3·1 we have defined *the* **centre of pressure** *to be the point of the plane area at which the resultant of this system of parallel fluid thrusts acts.*

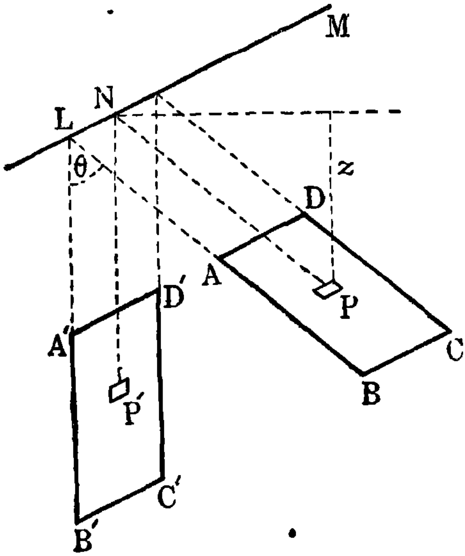
If the plane area exposed to the fluid thrust be divided into elements of area $\alpha_1$, $\alpha_2$,......, whose depths below the surface of zero pressure are $z_1$, $z_2$,......, then the thrusts on all the elements are given by $\Sigma w\alpha_1 z_1$, where $w$ denotes the weight of unit volume of the liquid. As shown in 2·81, the *magnitude* of the resultant of the thrusts $\Sigma w\alpha_1 z_1$ is $wS\bar{Z}$, where $S$ is the area of the portion of the plane surface in contact with the liquid and $\bar{Z}$ the depth of the centre of gravity of $S$. The *direction* of this resultant thrust or whole pressure, is of course normal to the plane area; the problem now is to determine the particular *point* in the plane area at which this resultant $wS\bar{Z}$ acts.

In 3·11 we have already stated without proofs the position of the centre of pressure, or C. P., for certain simple cases. Their proofs will be given now and the general question of the determination of centre of pressure will be systematically treated. But before we enter into details, we make an important observation which will prove quite useful in subsequent discussions.

**4·11. Position of the C. P. unaltered by rotation of its plane area.** *If the plane of an area in contact with a*

*liquid be turned about its line of intersection with the effective surface, the position of its centre of pressure relative to the area remains unaltered.*

Suppose $ABCD$ is a plane area inclined at an angle $\theta$ to the vertical and let $LM$ be its line of intersection with the effective surface. Let the plane $ABCD$ be rotated about $LM$ until it becomes vertical, so that $A'$, $B'$, $C'$ and $D'$ become the new positions of $A$, $B$, $C$ and $D$ respectively.



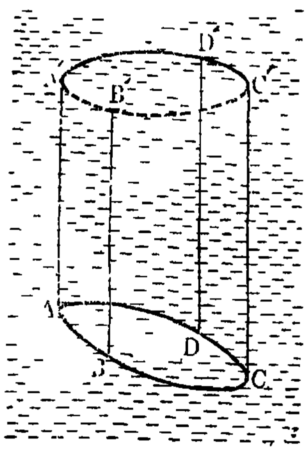

If $P$ denotes an element of area $\alpha$ at depth $z$, the new depth of $\alpha$ being $P'N$, is equal to $z \sec\theta$. Since the pressure at a point is proportional to its depth, the effect of the rotation is that the thrust on each element of area is multiplied by $\sec\theta$ and their directions are all turned through the same angle.

Now we know from the principles of the Composition of Parallel Forces that if every force of a system of parallel forces be changed in the same ratio, the relative position of the point where their resultant acts, remains unaltered. This proves the statement.

4·12. **Remark.** It must be clear from above that *when for any given plane area our object is to find the position on the area of its centre of pressure, there is absolutely no loss of generality in supposing the plane of the area to be at any inclina-*

*tion or, in particular, to be vertical.* This may afford a good deal of simplicity and convenience in finding the position of the C P. relative to the area.

**4·2. Geometrical method for finding C. P.** Suppose a plane area $ABCD$, not in a vertical position, is immersed in a liquid. From every point of the perimeter of the area conceive vertical lines drawn to meet the surface of zero pressure in the curve $A'B'C'D'$.




Considering the equilibrium of the superincumbent liquid we find that the only vertical forces on it are its weight acting downwards through its centre of gravity and the vertical component of the reaction of the plane area $ABCD$ upon the liquid acting upwards at the point of the centre of pressure of the area. Since these two balance one another, we conclude that the vertical line passing through the centre of gravity of the superincumbent liquid meets the plane area at the point of its centre of pressure.

Also since for a homogeneous liquid the pressure on an element of the immersed plane area is equal to the weight of the column of liquid standing on it, and the C. G. of this column is at half the depth of the element of area, it is evident that the depth of the C. G. of the whole superincumbent liquid will be one half of the depth of the C. P. of the plane area.

When the plane of the area $ABCD$ is vertical, there will be no superincumbent liquid and the foregoing

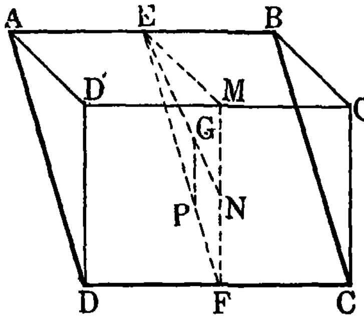
method may seem to fail. But the area may be conveniently rotated about its line of intersection with the effective surface and the C. P. of the area in this inclined position may be determined by the method just explained. In view of the result of 4·11, the same point would give the position of the C. P. of the area even when its plane is vertical.

Thus we have the following rule :—

**The centre of pressure of a plane area in contact with a liquid is the point of the area in which it is met by the vertical line drawn through the centre of gravity of the superincumbent liquid.**

4·21. **Application.** We illustrate the application of the above method by finding the centre of pressure of a rectangular plane area.

Suppose a rectangle *ABCD* with its side *AB* in the surface of the liquid is inclined at a finite angle to the vertical plane through *AB*.




Drawing verticals through all the points on *AD*, *DC*, *CB* to meet the surface in *AD′*, *D′C′*, *C′B* respectively, we get the superincumbent liquid in the shape of a triangular prism *ABC′D′DC*.

If *E*, *F*, *M* be the middle points of the sides *AB*, *CD*, *C′D′* respectively, then it is clear that the C. G. of the superincumbent liquid coincides with the C. G. of the $\triangle EFM$. Let this C. G. be the point *G* on the line *EN*, where *N* is the middle point of *FM*.

From *G* draw a vertical line which meets *EF* at *P*. Then *P* is the centre of pressure of the rectangular area *ABCD*.

Now $GP$ and $NF$, being both vertical, are parallel to each other. Hence we have

$$EP : EF = EG : EN = 2 : 3.$$

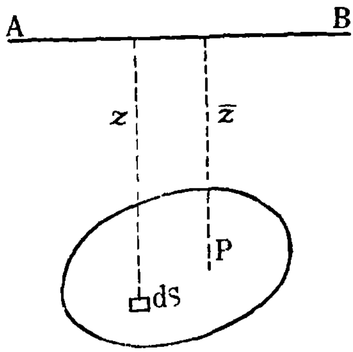
Therefore

$$EP = \tfrac{2}{3}\, EF.$$

In view of 4·11, the position of C. P. will remain unaltered even when its plane is vertical.

Thus *the C. P. of a rectangle ABCD with the side AB in the surface is at a point P situated on EF such that* $EP = \frac{2}{3}EF$, *E and F being the middle points of the sides AB and CD respectively.*

**4·3 Formulae for the depth of the C. P.** Let the given plane area be vertical and let it meet the surface of zero pressure in the line $AB$. Let $ds$ denote an element of area at depth $z$ from $AB$. Suppose the centre of pressure of the plane area is at the point $P$ whose depth is denoted by $Z$.




If $\rho$ be the density of the liquid, then the thrust on $ds$ is $g\rho z\,ds$. The total thrust is the resultant of the system of parallel forces of the type $g\rho z\,ds$, and acts at the centre of pressure of the area.

Now taking moments about the line $AB$ we get

$$Z \times \Sigma g\rho z\,ds = \Sigma(g\rho z\,ds \times z)$$

or

$$Z = \frac{\Sigma g\rho z^2 ds}{\Sigma g\rho z\,ds},$$

or

$$\mathbf{Z} = \frac{\Sigma\, \mathbf{z}^2\mathbf{ds}}{\Sigma\, \mathbf{zds}}. \qquad \ldots \ldots \quad (1)$$

Using the notation of the *Integral Calculus*, the depth $\bar{z}$ of the centre of pressure obtained above, may be given by

$$\bar{z} = \frac{\int z^2\, ds}{\int z\, ds}. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

If the element of area $ds$ be expressed as $dx\, dy$, then the formula (2) may be written as

$$\bar{z} = \frac{\iint z^2\, dx\, dy}{\iint z\, dx\, dy}. \quad . \quad . \quad . \quad . \quad (3)$$

When the pressure is not necessarily proportional to $z$, the depth of the C. P.

$$\bar{z} = \frac{\iint zp\, dx\, dy}{\iint p\, dx\, dy}, \quad . \quad . \quad . \quad . \quad (4)$$

where $p$ is the pressure at depth $z$.

Evidently the integrations in (2), (3) and (4) are to be so performed that the whole of the given plane area is covered.

By turning the plane area about $AB$, the relative position on the area of its centre of pressure remains unaltered [4·11]. Hence the results obtained above also hold for any inclined position of the plane, provided that in this case instead of representing 'depths', $z$ represents distances from $AB$ measured along the line of greatest slope on the plane.

**4·31. Application to Standard Cases.** In 3·11 the position of the centre of pressure was given without proofs for four standard cases. Their proofs can be given in more than one way, but for the sake of simplicity and elegance we prefer to establish these results by the application of the formulae obtained in the preceding section.

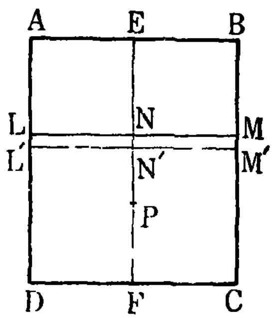
As remarked in 4·12, there is no loss of generality in taking the plane of the area to be vertical. Hence, although while establishing the proofs, the plane, for the sake of simplicity, is taken to be vertical, the results arrived

at will be equally applicable even when the plane of the area is oblique.

In what follows the atmospheric pressure is neglected, but whenever required, its effect can be easily taken into account.

**(i) Centre of pressure of a rectangular area immersed in a homogeneous liquid with one side in the surface.**

Suppose the side $AB$ is in the surface of the liquid. Let $AB = a$ and $AD = b$, and let $E$ and $F$ be the middle points of $AB$ and $DC$ respectively.




It is clear from symmetry that the C. P. will lie on $EF$.

Let the depth $z$ from $AB$ be measured along $EF$. Supposing the area to be divided into elementary strips parallel to $AB$, let $LM\,M'L'$ be one of such strips so that $AL = z$ and $LL' = dz$.

If $\rho$ be the density of the liquid, the pressure on the strip $LMM'L'$ is $g\rho z a dz$ acting at distance $z$ from $AB$. Hence, if $Z$ denotes the distance of the centre of pressure $P$ from $AB$, we have

$$Z = \frac{\Sigma g\rho z a dz \times z}{\Sigma g\rho z a dz} = \frac{\Sigma z^2 dz}{\Sigma z dz}$$

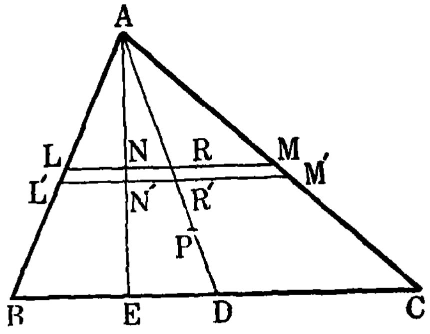
$$= \frac{\int_0^b z^2 dz}{\int_0^b z dz} = \frac{\left[\frac{z^3}{3}\right]_0^b}{\left[\frac{z^2}{2}\right]_0^b},$$

or $$\mathbf{Z = \tfrac{2}{3}\, b = \tfrac{2}{3}\, EF.}$$

**Remark.** The above proof and the result will also hold if the rectangle be replaced by a parallelogram with one side in the surface.

**(ii) Centre of pressure of a triangular area immersed in a liquid with its vertex in the surface and base horizontal.**

Let $BC = a$ and $D$ the middle point of $BC$. Suppose $AD = \gamma$.




Let the triangular area $ABC$ be divided by horizontal elementary strips like $LMM'L'$. The thrusts on all such strips will act at their middle points which lie on the median $AD$. Hence the centre of pressure of the triangle will lie on $AD$, say at the point $P$. Let $AE$ be the perpendicular on $BC$, and let $AR = z$ and $RR' = dz$.

Now

$$\frac{LM}{BC} = \frac{AR}{AD} = \frac{z}{\gamma}.$$

$$\therefore \qquad LM = za/\gamma. \qquad \ldots \ldots (1)$$

Also
$$\frac{AN}{AR} = \frac{NN'}{RR'} = \frac{AE}{AD} = K. \qquad \ldots (2)$$

If $w$ be the specific weight of the liquid, the thrust on the strip $LM'$

$$= w \times AN \times LM.NN'$$
$$= w \times K.AR \times (a/\gamma)z.K.RR', \text{ from (1) and (2)}$$
$$= wK^2(a/\gamma)z^2dz.$$

Now, if $AP = \bar{Z}$, we have

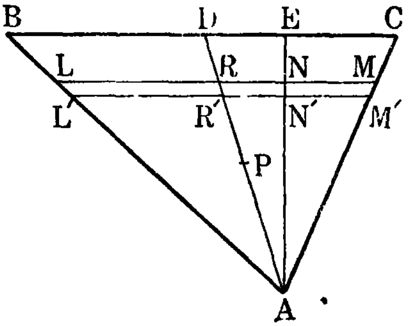
$$\bar{Z} = \frac{\Sigma\left(\frac{wK^2a}{\gamma}\right)z^2\,dz \times z}{\Sigma\left(\frac{wK^2a}{\gamma}\right)z^2\,dz}$$

$$=\frac{\int_0^\gamma z^3\,dz}{\int_0^\gamma z^2\,dz}=\frac{\left[\frac{z^4}{4}\right]_0^\gamma}{\left[\frac{z^3}{3}\right]_0^\gamma},$$

or $$\bar{Z}=(3/4)\,\gamma=(3/4)\,AD.$$

**(iii) Centre of pressure of a triangular area immersed in a liquid with one side in the surface.**

In the adjoining figure let $BC=a$, $D$ be the middle point of $BC$ and $AD=\gamma$.




Suppose the triangular area $ABC$ is divided into horizontal elementary strips of the type $LMM'L'$. Then it is clear that the centre of pressure of $ABC$ will lie on $AD$. Let this be the point P. Let $DR=z$, $RR'=dz$ and $DA=\gamma$.

Here

$$LM=BC\times\frac{AR}{AD}=a\,\frac{\gamma-z}{\gamma}.$$

As in the case (ii), it is seen that the pressure on the strip $LM'$ is proportional to

$$wz\times\{a(\gamma-z)/\gamma\}\,dz=\lambda(\gamma-z)z\,dz,\text{ say.}$$

Hence, if the distance of the centre of pressure $DP=\bar{Z}$, we get

$$\bar{Z}=\frac{\Sigma\lambda(\gamma-z)z\,dz\times z}{\Sigma\lambda(\gamma-z)z\,dz}$$

$$=\frac{\int_0^\gamma(\gamma-z)z^2dz}{\int_0^\gamma(\gamma-z)z\,dz}=\frac{\left[\gamma\frac{z^3}{3}-\frac{z^4}{4}\right]_0^\gamma}{\left[\gamma\frac{z^2}{2}-\frac{z^3}{3}\right]_0^\gamma}$$

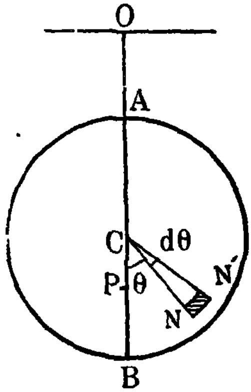
$$=\frac{\gamma^4/3-\gamma^4/4}{\gamma^3/2-\gamma^3/3},$$

or $$\bar{Z}=(1/2)\,\gamma=(1/2)\,\text{DA}.$$

**(iv) Centre of pressure of a vertical circular area of radius a, totally immersed in a liquid with its centre at depth h below the surface.**

It will be found more convenient to use polar coordinates in this case.

Let $C$ be the centre. Take a point $N$ on the area whose polar coordinates referred to the centre of the circle and the downward vertical radius be $r$ and $\theta$.




From symmetry it is clear that the C. P. will be on the vertical line $OCB$, say at the point $P$.

The area of the small element of surface at $N$ is $r\,\delta r\,\delta\theta$, and its depth below the surface of the liquid is $h+r\cos\theta$.

Hence the pressure on this area

$$=w(r\,\delta r\,\delta\theta)(h+r\cos\theta),$$

and it acts at a point whose depth from the horizontal through the centre is $r\cos\theta$.

Therefore the depth of the C. P. of the circle from the centre

$$=\frac{2\int_0^a\int_0^\pi r(h+r\cos\theta)r\cos\theta\,dr\,d\theta}{2\int_0^a\int_0^\pi r(h+r\cos\theta)dr\,d\theta}$$

$$=\frac{\int_0^a\left[r^2h\sin\theta+\frac{1}{2}r^3\theta+\frac{1}{4}r^3\sin 2\theta\right]_0^\pi dr}{\int_0^a\left[rh\theta+r^2\sin\theta\right]_0^\pi dr}$$

$$= \frac{\int_0^a \frac{1}{2} r^3 \pi \, dr}{\int_0^a r h \pi \, dr} = \tfrac{1}{8} a^4 \pi \div \tfrac{1}{2} a^2 h\pi$$

$$= (1/4)\, \mathbf{a^2/h}.$$

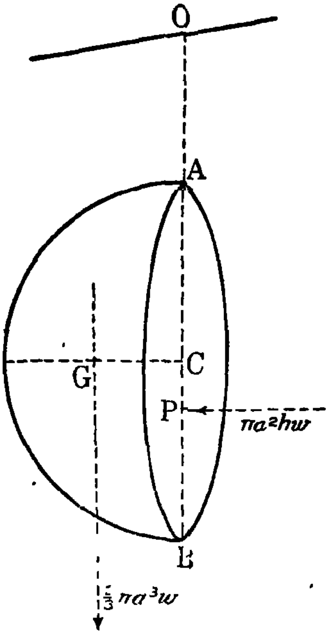
*Thus the depth of the C. P. from the surface of the liquid*

$$\mathbf{OP = h + \tfrac{1}{4} a^2/h}.$$

**Alternative Method**

This result can be established in a simpler manner as follows :—

Let us consider the equilibrium of a hemispherical mass of liquid of which the circular area of radius $a$ forms the base. The weight of this mass of liquid is $\frac{2}{3}\pi a^3 w$ and acts downwards through the centre of gravity $G$; this is balanced by the thrusts on its bounding surface which is partly curved and partly plane. The thrusts on the curved surface all pass through the centre $C$ and so must their resultant. The resultant thrust on the circular plane is $\pi a^2 h w$ and acts through the centre of pressure $P$ perpendicular to its plane.




Now taking moments about $C$, we get

$$\pi a^2 h w \times CP = \tfrac{2}{3}\pi a^3 w \times CG = \tfrac{2}{3}\pi a^3 w \times \tfrac{3}{8} a,$$

or
$$\mathbf{CP = a^2/4h}.$$

**Corollary.** When the circular area is inclined, say at an angle $\theta$ to the horizontal, the same formula by 4·11

will represent the distance $CP$ provided that $h$ denotes the slant distance of $C$ from the surface measured along the plane of the circle. If, however, $h$ is taken to denote the *vertical depth* of $C$, then since its distance along the inclined plane of the circle will become $h \operatorname{cosec} \theta$, the distance $CP$ will be given by

$$\mathbf{CP} = \tfrac{1}{4}\,(\mathbf{a}^2/\mathbf{h})\,\mathbf{sin}\;\theta\,.$$

The above result for the case of inclined circular plane can be obtained directly by equating moments of $X$ and the weight $\frac{2}{3}\, g\varrho\pi a^3$ of the liquid about the point $O$ in the example (i) of 3·69. For, from the figure annexed to that example, we get

$$X \cdot OP = \tfrac{2}{3}\, g\varrho\, \pi a^3 w \times OG \sin\theta,$$

or $$g\varrho\pi a^2 hw \times OP = \tfrac{2}{3}\, g\varrho\pi a^3 w \times \tfrac{3}{8}\, a \sin\theta,$$

or $$\mathbf{OP} = \tfrac{1}{4}\,(\mathbf{a}^2/\mathbf{h})\,\mathbf{sin}\;\theta.$$

**4·4. Analogy in the determination of C .P. and C. G.** The C. P. being the centre of a system of parallel forces consisting of fluid thrusts, the methods for its determination are naturally analogous to those used for the determination of the centre of gravity. Almost exactly as we proceed in *Statics* for finding C. G., we can treat cases of the following types for finding C. P. :—

(i) *If the position of the centre of pressure and thrust on each of the two portions into which a given area is divided be known, to find the position of the C. P. of the whole area.*

(ii) *If the C. P. of and the thrusts on, the whole area and one of its two parts into which the area is divided be known, to find the C. P. of the other part.*

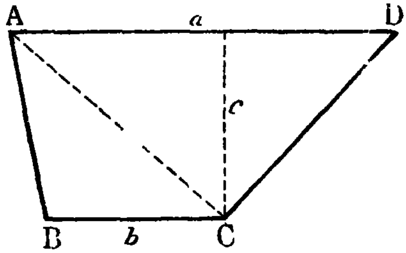
(iii) *To find the C. P. of a given area when it can be expressed as a combination of several standard types of areas for which the C. P. are known.*

**4·41. Illustrative Example.** *Prove that the depth of the centre of pressure of a trapezium immersed in water with the side 'a' in the surface and the parallel side 'b' at a depth c below the surface is*

$$\left(\frac{a+3b}{a+2b}\right)\frac{c}{2}$$

[*Calcutta*, 1912, *Lucknow*, 1928, 1934, 1940, *Agra*, 1934]

Let $ABCD$ be the trapezium. Divide the trapezium into two triangles by joining the points $A$, $C$.




If $w$ be the weight of unit volume of the liquid, the thrust on the $\triangle ABC$

$$= w(bc/2)(\tfrac{2}{3}c) = \tfrac{1}{3}w\,bc^2,$$

and it acts, as shown in (ii) of **4·31**, at a depth $\frac{3}{4}c$ below $AD$.

Again, the thrust on the $\triangle ACD$

$$= w(ac/2)(c/3) = \tfrac{1}{6}wac^2,$$

and it acts, as proved in (iii) of **4·31**, at a depth $\frac{1}{2}c$ below $AD$.

Hence, the depth of the C. P. of the trapezium $ABCD$

$$= \frac{\frac{1}{3}w\,bc^2 \times \frac{3}{4}c + \frac{1}{6}wac^2 \times \frac{1}{2}c}{\frac{1}{3}w\,bc^2 + \frac{1}{6}wac^2}$$

$$= \tfrac{1}{2}c\left(\frac{b/2 + a/6}{b/3 + a/6}\right)$$

$$= \left(\frac{a+3b}{a+2b}\right)c/2$$

EXAMPLES VII

1. Prove that the centre of pressure of a triangle $ABC$ immersed in a homogeneous liquid with the side $BC$ in the surface, coincides with the centre of the two equal forces each $\frac{1}{6}w\Delta\alpha$ acting at the middle points of $AB$ and $AC$, where $\alpha$ is the vertical depth of $A$ below $BC$, $\Delta$ the area of the triangle and $w$ the specific wt. of the liquid.

[*Allahabad*, 1931]

2. The gate of a lock is 10 ft. wide and 18 ft. deep and it has the pressure due to 15 ft. of fresh water on one side and 10 ft. of sea water on the other side. Find the magnitude and position of the resultant pressure on the surface of the gate.

[Fresh water weighs 62·3 lbs. per cu. ft. and sea water weighs 64 lbs. per cu. ft.] [*Benares, Eng.*, 1938]

3. In the vertical side of a vessel containing water there is a square trap-door, opening freely outwards about a hinge in its upper edge which is horizontal. The length of a side of the door is 3 cm. and water rises in the vessel upto the level of the hinge. Find the least force (in grammes) that will keep the trap-door closed.

4. A triangle is wholly immersed in a liquid with its base in the surface. Show that a horizontal straight line drawn through the centre of pressure of the triangle divides it into two parts, the pressures on which are equal. [*Calcutta*, 1937]

5. Prove that the horizontal line through the centre of pressure of a rectangle immersed in a liquid with one side in the surface, divides the rectangle in two parts, the fluid pressures on which are in the ratio 4 : 5.

6. A rectangle is immersed vertically in a heavy homogeneous liquid with two of its sides horizontal and at depths $a$ and $b$ below the surface. Find the depth of the centre of pressure.

[*Calcutta*, 1918; *Patna*, 1942]

7. A lamina in the shape of a quadrilateral $ABCD$ has the side $CD$ in the surface, and the sides $AD$, $BC$ vertical and of lengths $\alpha$ and $\beta$ respectively. Prove that the depth of the centre of pressure is

$$\frac{3}{4}\,\frac{\alpha^4-\beta^4}{\alpha^3-\beta^3}.$$

8. Find the centre of pressure of a square lamina immersed in a fluid, with one vertex in the surface and the diagonal vertical.

[*Allahabad*, 1920]

9. A rhombus is immersed in a liquid with a vertex in the surface and the diagonal through the vertex vertical. Prove that the centre of pressure divides the diagonal in the ratio 7 : 5.

[*Nagpur*, 1943; *Agra*, 1927, *Allahabad*, 1931]

10. A quadrilateral is immersed in water with two angular points in the surface and the other two at depths $a$ and $b$. If $x$ and $y$ are the depths below the surface of its centres of gravity and pressure respectively, show that,

$$6xy + ab = 3x(a+b).$$ [*Allahabad*, 1926]

11. A quadrilateral $ABCD$, with sides $AB$, $CD$ parallel (and $AD$, $BC$ equal) is immersed in a homogeneous liquid with $AB$ in

the free surface. Show that the centre of pressure will be the intersection of the diagonals, if $AB = \sqrt{3}\, CD$. [*Bombay*, 1940]

12. $ABC$ is a triangular lamina immersed vertically in water with $C$ in the surface and $AB$ horizontal. Show how to divide the area by a horizontal line $PQ$ into two portions on which the pressures are equal, $P$ and $Q$ being points on $AB$ and $BC$ respectively.

If $h$ be the length of the perpendicular from $C$ on $AB$, prove that the height above $AB$ of the centre of pressure of the area $APQB$ in the above case is

$$\tfrac{1}{8}h(3\sqrt[3]{4} - 4).$$ [*Benares*, 1933]

## 4·5. Effect of further immersion.

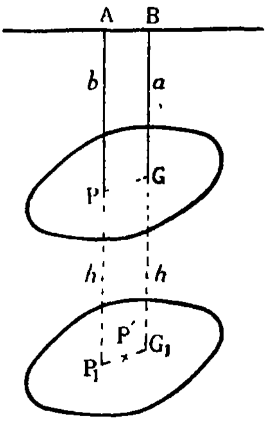
When a plane area immersed in a liquid is further lowered down, without rotation, the pressure on every element of the area will be increased and the C. P. will not remain in its original position unless the plane of the area is horizontal. In case the plane area is horizontal, the pressure on it everywhere will be the same and the position of C. P. and C. G. will be coincident; the effect of further lowering the area will be to increase uniformly the pressure on all elements without causing the position of C. P. and C. G. to separate. Let us now consider the problem created on account of further immersion of an area which is not necessarily horizontal.

**A plane area is immersed in a homogeneous liquid with its centres of gravity and pressure at depths a and b respectively; if the whole area is now lowered to a further depth h, without rotation, to find the new position of the centre of pressure of the area.**

Let $G$ and $P$ be the centres of gravity and pressure of the area $S$ at respective depths $a$ and $b$ from the surface $AB$. When the area $S$ is lowered to a depth $h$, let $P_1$, $G_1$ denote the points $P$ and $G$ of the area respectively.

The resultant thrust on the area in the original position has been a force $g\rho aS$ acting at $P$, where $\rho$ denotes the density of the liquid. The effect of the lowering of

the area, without rotation, to a further depth $h$, is to increase the pressure at every point of the area by the same amount which is $g\rho h$.




Since this extra pressure is *uniform* over the whole area, its resultant in the new position of the area, viz., $g\rho hS$, will act at $G_1$.

Hence in the lowered position of the area there are two thrusts—the previous thrust $g\rho aS$ acting at $P_1$ and the additional thrust $g\rho hS$ acting at $G_1$. Their resultant thrust is $g\rho(a+h)S$ acting at a point $P'$ on $P_1G_1$, which is the new position of the centre of pressure of the area.

By the rule of compounding parallel forces, we have

$$P_1P' : P'G_1 :: g\rho hS : g\rho aS,$$

or

$$\mathbf{P_1P' : P'G_1 = h : a.} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

If $Z$ denotes the depth of $P'$ from $AB$, we have by taking moments about $AB$

$$Z = \frac{g\rho aS(b+h) + g\rho hS(a+h)}{g\rho aS + g\rho hS},$$

or

$$Z = \frac{\mathbf{h^2 + 2ah + ab}}{\mathbf{a+h}}. \quad . \quad . \quad . \quad (2)$$

**4·51. Corollaries.** From the foregoing discussions certain useful results can be easily deduced.

(i) We have

$$\text{depth of } P' - \text{depth of } P_1$$
$$= \frac{h^2 + 2ah + ab}{a+h} - (b+h)$$
$$= -\frac{b-a}{h+a} \cdot h,$$

which is *always* negative.

Hence, as a consequence of further immersion, the centre of pressure in the area itself is raised by a distance

$$\frac{b-a}{b+a} \cdot h .$$

(ii) We have

the depth of $P'$ − depth of $G_1$

$$= \frac{h^2 + 2ah + ab}{a+h} - (a+h)$$

$$= \frac{ab - a^2}{a+h},$$

which varies inversely as $a + h$, that is, as the depth of the C G.

It follows that the greater the value of $h$, the smaller will be the distance between $P'$ and $G_1$, so much so that when the depth by which the area is lowered tends to infinity, the centre of pressure coincides with the centre of gravity in the limit.

This is also evident from the fact that when $h$ is indefinitely large, the thrust $g\varrho aS$ acting at $P_1$ is negligibly small compared to the thrust $g\varrho hS$ acting at $G_1$, and hence the point where the resultant of these two thrusts acts, i.e., the centre of pressure, will approach $G_1$ in measure with the comparative greatness of $g\varrho hS$ over $g\varrho aS$.

(iii) If the centre of pressure of a given area be known when the atmospheric pressure is neglected, its position may be easily determined when the atmospheric pressure is taken into account.

For, by 4·5, the effect of taking into account an atmospheric pressure $\Pi$ is equivalent to supposing that the given area has been further lowered by a depth $\Pi/g\varrho$, because a height $\Pi/\varrho g$ of that liquid would produce the same pressure $\Pi$ as the atmosphere.

**Note.** In working out examples sometimes it may prove convenient to suppose that instead of the area being

lowered through a depth $h$, a depth $h$ of the liquid has been superimposed on the surface of the liquid.

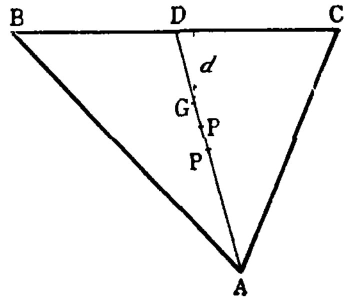
**4·6. Illustrative Example.** *A triangle has its base in the surface of a liquid and its vertex downwards; if the atmospheric pressure be equivalent to a height h of water, prove that the centre of pressure will be higher by a distance*

$$\tfrac{1}{2}\frac{hd}{h+d}$$

*than it is when the atmospheric pressure is neglected, where d is the distance of the centre of gravity of the triangle below the surface of the water.*

[*Lucknow*, 1932; *Allahabad*, 1938]

Let $G$ be the centre of gravity of the triangle $ABC$, with its base $BC$ in the surface of the liquid, and $P$ its centre of pressure when the atmospheric pressure is neglected.




Since $d$ is the depth of $G$ below $BC$, the depth of the vertex A is $3d$, and $P$ being at the middle point of the median $AD$—case (iii) of 4·31—its depth is $\frac{3}{2}d$. Let $\Delta$ be the area of the triangle and $w$ the specific weight of the liquid.

When the atmospheric pressure is taken into account, the following are the thrusts acting on the triangle :—

(1) The whole pressure $w\Delta d$ acting at $P$, i.e., at depth $\frac{3}{2}d$ below $BC$.

(2) The additional thrust $w\Delta h$ due to the atmospheric pressure being taken into account; this acts at $G$ which is at depth $d$ from $BC$.

Now, if $P'$ be the new position of the centre of pressure, its depth from $BC$

$$=\frac{w\Delta d\times\frac{3}{2}d+w\Delta h\times d}{w\Delta d+w\Delta h}$$

$$=\left(\frac{3d+2h}{d+h}\right)\frac{d}{2}.$$

Hence, the vertical distance between $P$ and $P'$

$$=\frac{3d}{2}-\frac{3d+2h}{d+h}\cdot\frac{d}{2}.$$

$$=\tfrac{1}{2}\frac{hd}{d+h}.$$

## Examples VIII

1. Find the centre of pressure of an isoceles triangle immersed with its plane vertical and its base horizontal half as far below the surface as its vertex. [*Calcutta*, 1911]

2. A triangle is immersed in a liquid with the base horizontal and vertex in the surface. If the atmospheric pressure is equivalent to a head of H feet of the liquid, find through what height the centre of pressure of the triangle is raised in the plane of the triangle. [*Bombay*, 1935]

3. An equilateral triangle $ABC$ of altitude $h$ has one vertex fixed at a depth of $2h$ below the surface of a liquid. From the position in which $BC$ is horizontal above $A$, the triangle is turned till $BC$ is again horizontal but below $A$, the plane of the triangle always remaining vertical. Show that the shift of the centre of pressure relative to the triangle, is $h/16$. [*Agra*, 1938]

4. A square lamina is just immersed vertically in water and is then lowered through a depth $h$, if $a$ is the length of the edge of the square, prove that the distance of the centre of pressure from the centre of the square is $a^2/(6a + 12h)$. [*Allahabad*, 1935]

5. A rectangle is immersed vertically in water with two sides horizontal and at depths $h$ and $a + h$ respectively below the effective surface; prove that the distance of the C. P. from the upper side is

$$\frac{a}{3}\left(\frac{3h + 2a}{2h + a}\right).$$

6. A rectangular area of a height $h$ is immersed vertically in a liquid with one side in the surface; show how to draw a horizontal line across the area so that the centres of pressure of the parts of the area above and below this line of division shall be equally distant from it.

7. A parallelogram $ABCD$ is immersed in a liquid with A in the surface, and $BD$ horizontal. Prove that the centre of pressure $P$ lies on $AC$ such that $AP \,.\, AC = 7 : 12$. [*Patna*, 1935]

8. A quadrilateral is immersed vertically having two sides of lengths $2a$, $a$ parallel to the surface at depths $h$ and $2h$ respectively. Show that the depth of the centre of pressure is $3h/2$. [*M. T.*]

9. When the depth of the liquid is increased by an amount $a$, the centre of pressure is found to increase by $y$, and when instead, the depth of the liquid is increaed by $b$, that of the centre of pressure is found to increase by $z$. Show that the depth of the centre of

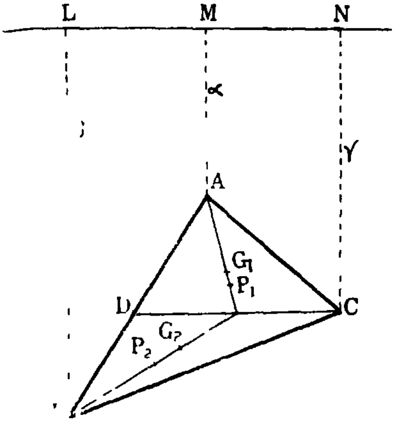
gravity of the area in the original state of the liquid is

$$ab\,(b-a+y-z)/(az-by).$$

10. A plane area is completely immersed in water, its plane being vertical, it is made to descend in a vertical plane without any rotation and with uniform velocity; show that the centre of pressure approaches the horizontal through its centre of mass with a velocity which is inversely proportional to the square of the depth of its centre of mass.

**4·7. Depth of the C. P. of a triangle in terms of the depths of its three vertices.** Let $ABC$ be a triangle with its vertices $A$, $B$, $C$ at depths $\alpha$, $\beta$, $\gamma$ respectively.

Through C draw a horizontal line $CD$ to divide the given triangle into two triangles $ACD$ and $BCD$.




Knowing the thrusts and the centres of pressure of the triangles $ACD$ and $CBD$ from (ii) and (iii) of 4·11, and also from 4·5 the additional thrusts on account of their immersion through depths $\alpha$ and $\gamma$ respectively, we have the thrust on the given triangle $ABC$ which consists of the following:—

I. The thrust $w\times\frac{1}{2}CD(\gamma-\alpha)\times\frac{2}{3}(\gamma-\alpha)$ on the $\triangle ACD$ acting at its centre of pressure $P_1$, i.e., at depth $\frac{3}{4}(\gamma-\alpha)+\alpha$ $(=\frac{1}{4}(3\gamma+\alpha))$ from the surface $LMN$.

II. The additional thrust on account of further immersion of the $\triangle ACD$, viz., $w\times\frac{1}{2}CD(\gamma-\alpha)\times\alpha$ acting at the centre of gravity $G_1$ at depth $\frac{2}{3}(\gamma-\alpha)+\alpha$ $(=\frac{1}{3}(2\gamma+\alpha))$ from $LMN$.

III. The thrust $w\times\frac{1}{2}CD(\beta-\gamma)\times\frac{1}{3}(\beta-\gamma)$ on the $\triangle BCD$ acting at its centre of pressure $P_2$ at depth $\frac{1}{2}(\beta-\gamma)+\gamma$ $(=\frac{1}{2}(\beta+\gamma))$ from $LMN$.

IV. The additional thrust $w \times \frac{1}{2}CD(\beta - \gamma) \times \gamma$ on account of immersion acting at $G_2$ at depth $\frac{1}{3}(\beta - \gamma) + \gamma$ $(= \frac{1}{3}(2\gamma + \beta))$ from $LMN$.

We have, therefore, the depth of the centre of pressure of the $\triangle ABC$ from $BC$

$$= \frac{\begin{array}{l}[\frac{1}{6}w.CD.(\gamma-\alpha)^2\times\frac{1}{4}(3\gamma+\alpha)+\frac{1}{2}w.CD(\gamma-\alpha)\alpha\times\frac{1}{3}(2\gamma+\alpha)\\ \quad+\frac{1}{6}w.CD(\beta-\gamma)^2\times\frac{1}{2}(\beta+\gamma)+\frac{1}{2}w.CD(\beta-\gamma)\gamma\times\frac{1}{3}(2\gamma+\beta)]\end{array}}{\begin{array}{r}[\frac{1}{6}w.CD(\gamma-\alpha)^2+\frac{1}{2}w.CD(\gamma-\alpha)\alpha+\frac{1}{6}w.CD(\beta-\gamma)^2\\ +\frac{1}{2}w.CD(\beta-\gamma)\gamma]\end{array}}$$

$$= \frac{\begin{array}{r}\frac{1}{2}[(\gamma-\alpha)^2(3\gamma+\alpha)+2\alpha(\gamma-\alpha)(2\gamma+\alpha)+(\beta-\gamma)^2(\beta+\gamma)+\\ 2\gamma(\beta-\gamma)(2\gamma+\beta)]\end{array}}{[2(\gamma-\alpha)^2+3\alpha(\gamma-\alpha)+(\beta-\gamma)^2+3\gamma(\beta-\gamma)]} \qquad \ldots\ldots\ldots \quad (1)$$

$$= \frac{1}{2}\,\frac{\alpha^2+\beta^2+\gamma^2+\beta\gamma+\gamma\alpha+\alpha\beta}{\alpha+\beta+\gamma}, \qquad \ldots\ldots\ldots \quad (2)$$

after making the necessary simplifications. For, the expression in the square brackets in the numerator of (1)

$$=(\gamma-\alpha)(3\gamma^2+2\alpha\gamma+\alpha^2)+(\beta-\gamma)(\beta^2+3\gamma^2+2\beta\gamma)$$
$$=3\gamma^2(\beta-\alpha)+2\gamma\{(\beta^2-\alpha^2)-\gamma(\beta-\alpha)\}+(\beta^3-\alpha^3)-\gamma(\beta^2-\alpha^2)$$
$$=\gamma^2(\beta-\alpha)+\gamma(\beta^2-\alpha^2)+(\beta^3-\alpha^3)$$
$$=(\beta-\alpha)(\alpha^2+\beta^2+\gamma^2+\beta\gamma+\gamma\alpha+\alpha\beta).$$

Again, the denominator in (1)

$$=(\gamma-\alpha)(2\gamma+\alpha)+(\beta-\gamma)(\beta+2\gamma)$$
$$=(\beta^2-\alpha^2)+\gamma(2\gamma+\alpha-2\alpha-\beta-2\gamma+2\beta)$$
$$=(\beta-\alpha)(\alpha+\beta+\gamma).$$

Thus the following important proposition has been established :—

**The depth of the centre of pressure of a triangular area whose vertices are at depths $\alpha$, $\beta$, $\gamma$, is given by**

$$\frac{1}{2}\,\frac{\alpha^2+\beta^2+\gamma^2+\beta\gamma+\gamma\alpha+\alpha\beta}{\alpha+\beta+\gamma}.$$

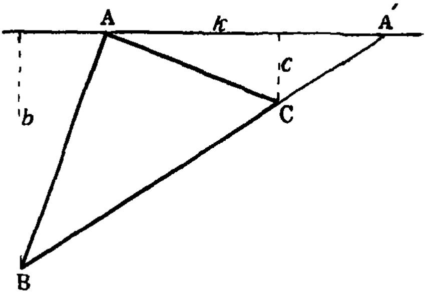
4·71. **Corollary.** If one of the vertices of the triangle, say $A$, is in the surface of the liquid, then $\alpha = 0$. Hence the rule:

**The depth of the centre of pressure of a triangle with a vertex in the surface and the other two vertices at depths $\beta$ and $\gamma$, is given by**

$$\tfrac{1}{2}\frac{\beta^2+\gamma^2+\beta\gamma}{\beta+\gamma}.$$

4·72. **Alternative method.** It is possible to prove the result given above in the corollary in an independent manner, and then to deduce from it the general result of 4·7 with the help of the theorem of 4·5.

Let $ABC$ be a triangle with the vertex $A$ in the surface. Let $BC$ produced meet the surface in $A'$. Let the depths of $B$ and $C$ be $b$ and $c$ respectively, and let $AA' = k$.




The thrust on the $\triangle\, AA'B$

$= w.\,(bk/2)\,.\,b/3$ acting at depth $b/2$ below $AA'$.

The thrust on the $\triangle\, AA'C$

$= w.\,(ck/2)\,.\,c/3$ acting at depth $c/2$ below $AA'$.

Hence, the depth of the C. P. of the $\triangle\, ABC$ from the surface

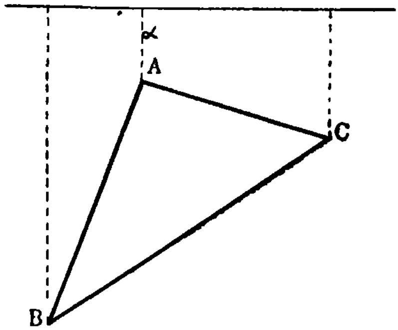
$$= \frac{\frac{1}{6}wb^2k \times b/2 - \frac{1}{6}wc^2k \times c/2}{\frac{1}{6}wb^2k - \frac{1}{6}wc^2k}$$

$$= \tfrac{1}{2}\frac{b^3-c^3}{b^2-c^2}$$

$$= \tfrac{1}{2}\frac{\mathbf{b}^2+\mathbf{bc}+\mathbf{c}^2}{\mathbf{b}+\mathbf{c}}. \qquad \text{(1)}$$

Suppose now that the triangle has been lowered through a depth $\alpha$, so that an additional thrust $w\triangle\alpha$ will act at its C. G., $\triangle$ being the area of the triangle.

Hence the depth of the C. P. in the new position from the horizontal through $A$




$$= \frac{w\Delta\left(\frac{b+c}{3}\right)\frac{1}{2}\left(\frac{b^2+bc+c^2}{b+c}\right) + w\Delta\alpha\left(\frac{b+c}{3}\right)}{w\Delta\left(\frac{b+c}{3}\right) + w\Delta\alpha}$$

$$= \frac{b^2+bc+c^2+2\alpha(b+c)}{2(3\alpha+b+c)} \quad . \quad . \quad . \quad (2)$$

Denoting by $\beta$ and $\gamma$ the depths of $B$ and $C$ in the lowered position, so that $\beta = b+\alpha, \gamma = c+\alpha$, we get from (2) the depth of the C. P. below the surface of the liquid

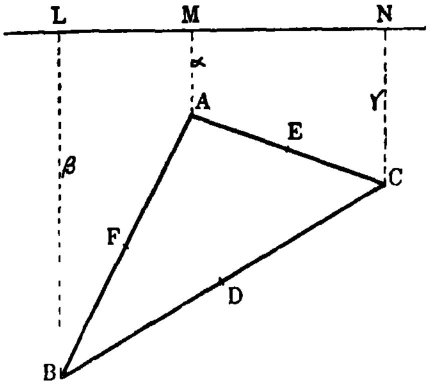
$$= \alpha + \frac{(\beta-\alpha)^2+(\beta-\alpha)(\gamma-\alpha)(\gamma-\alpha)^2 + 2\alpha(\beta+\gamma-2\alpha)}{2(\alpha+\beta+\gamma)}$$

$$= \frac{\alpha^2+\beta^2+\gamma^2+\beta\gamma+\gamma\alpha+\alpha\beta}{2(\alpha+\beta+\gamma)} \quad . \quad . \quad . \quad (3)$$

4·73. **Centre of Pressure of any triangle.** From the result of 4·7 the following important theorem can be deduced :—

**The centre of pressure of any triangular area, wholly immersed in a homogeneous liquid, coincides with the centre of three parallel forces acting at the middle points of the sides and of magnitudes proportional respectively to the depths of their middle points.**

The depth of the centre of pressure of a triangular area whose vertices are at depths $\alpha$, $\beta$, $\gamma$, is




$$= \frac{1}{2}\,\frac{\alpha^2+\beta^2+\gamma^2+\beta\gamma+\gamma\alpha+\alpha\beta}{\alpha+\beta+\gamma}$$

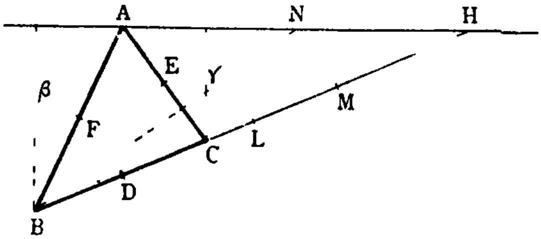
$$= \frac{\left[K\left(\frac{\beta+\gamma}{2}\right)\times\left(\frac{\beta+\gamma}{2}\right)+K\left(\frac{\gamma+\alpha}{2}\right)\times\left(\frac{\gamma+\alpha}{2}\right)+K\left(\frac{\alpha+\beta}{2}\right)\left(\frac{\alpha+\beta}{2}\right)\right]}{K\frac{\beta+\gamma}{2}+K\frac{\gamma+\alpha}{2}+K\frac{\alpha+\beta}{2}}, \quad . \; . \; . \quad (1)$$

where $K$ is a constant.

Since the depths of $D$, $E$, $F$, the middle points of $BC$, $CA$, $AB$, are $\frac{1}{2}(\beta+\gamma)$, $\frac{1}{2}(\gamma+\alpha)$, $\frac{1}{2}(\alpha+\beta)$ respectively, the expression (1) gives the vertical distance from $LMN$ of the centre of three parallel forces $K(\beta+\gamma)/2$, $K(\gamma+\alpha)/2$, $K(\alpha+\beta)/2$ which are proportional to the depths of $D$, $E$, $F$ respectively and act at these three middle points.

The above theorem can also be proved in a direct manner without making use of the result of 4·7. This is given in the next article.

**4·74. Second Proof.** We proceed to prove the theorem stated in the last article by making repeated use of the principles of compounding parallel forces.




Suppose in the first instance that the triangle $ABC$ has one of the its vertices, $A$, in the surface of the liquid, and $B$ and $C$ at depths $\beta$ and $\gamma$ respectively. Let $BC$ produced meet the surface at $H$ and suppose $D$, $E$, $F$, $L$, $M$ and $N$ are the middle points of $BC$, $CA$, $AB$, $BH$, $CH$ and $HA$ respectively. Let $AH = k$ and $w$ the specific weight of the liquid.

The total thrust on the triangle $ABH$

$= w.\frac{1}{2}\beta k.\beta/3$ acting at the middle point of the median $BN$

$= \frac{1}{12}wk\beta^2$ acting at $F + \frac{1}{12}wk\beta^2$ acting at L

$= \lambda\beta^2$ at $A + 2\ \lambda\beta^2$ at $B + \lambda\beta^2$ at $H$,

where $\lambda = \frac{1}{24}wk$.

Similarly, the total thrust on $CAH$

$= \lambda\gamma^2$ at $A + 2\ \lambda\gamma^2$ at $C + \lambda\gamma^2$ at $H$.

Hence, by subtraction, the thrust on the triangle $ABC$

$= \lambda(\beta^2 - \gamma^2)$ at $A + 2\lambda\beta^2$ at $B - 2\lambda\gamma^2$ at $C$
$+ \lambda(\beta^2 - \gamma^2)$ at $H$. . . (1)

Since $BH : CH :: \beta : \gamma$,

it is clear that the thrust at $H$, viz.

$$\lambda(\beta^2 - \gamma^2) = \lambda(\beta + \gamma)(\beta - \gamma),$$

may be considered as the resultant of

$-\lambda\gamma(\beta + \gamma)$ at $B$ and $\lambda\beta(\beta + \gamma)$ at $C$.

Therefore, the thrust on $ABC$ which is equivalent to (1),

$$\begin{aligned} &= \lambda(\beta^2 - \gamma^2), \text{ i.e., } \lambda(\beta-\gamma)(\beta+\gamma) \quad \text{at } A \\ &\quad + 2\lambda\beta^2 - \lambda\gamma(\beta+\gamma), \text{ i.e., } \lambda(\beta-\gamma)(2\beta+\gamma) \quad \text{at } B \\ &\quad + \lambda\beta(\beta+\gamma) - 2\lambda\gamma^2, \text{ i.e., } \lambda(\beta-\gamma)(\beta+2\gamma) \\ &\qquad \text{at } C. \quad \ldots\ldots\ldots \quad (2) \end{aligned}$$

Now if $\triangle$ denotes the area of the triangle $ABC$, so that

$$\triangle = \tfrac{1}{2}k\beta - \tfrac{1}{2}k\gamma,$$

we have

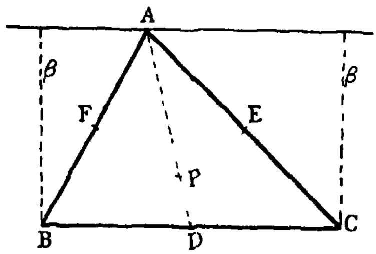
$$\begin{aligned} \lambda(\beta-\gamma) &= \tfrac{1}{24}kw(\beta-\gamma) = \tfrac{1}{12}w(\tfrac{1}{2}k\beta - \tfrac{1}{2}k\gamma) \\ &= \tfrac{1}{12}w\,\triangle. \end{aligned}$$

Hence, the thrust on $ABC$ which is equivalent to (2),

$$\begin{aligned} &= (w\triangle/12)(\beta+\gamma) \text{ at } A + (w\triangle/12)(2\beta+\gamma) \\ &\qquad \text{at } B + (w\triangle/12)(\beta+2\gamma) \text{ at } C \\ &= \{(w\triangle/12).\beta \text{ at } A + (w\triangle/12).\beta \text{ at } B\} \\ &\quad + \{(w\triangle/12).\gamma \text{ at } A + (w\triangle/12).\gamma \text{ at } C\} \\ &\quad + \{(w\triangle/12)(\beta+\gamma) \text{ at } B + (w\triangle/12)(\beta+\gamma) \\ &\qquad \text{at } C\} \\ &= (w\triangle/3).(\beta/2) \text{ at } F + (w\triangle/3).(\gamma/2) \text{ at } E \\ &\quad + (w\triangle/3).(\beta+\gamma)/2 \text{ at } D, \end{aligned}$$

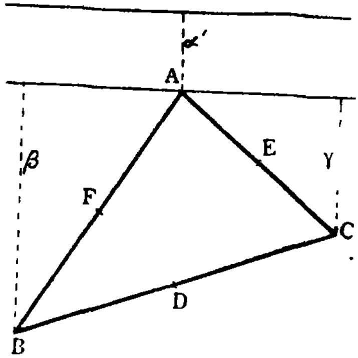
which are the forces acting at $D$, $E$, $F$, and of magnitudes proportional to their depths.

It may be noted that in the very special case when the side $BC$ of $ABC$ is horizontal, i.e., when $\gamma = \beta$, $BC$ produced will not meet the surface and the above proof may seem to fail. But knowing—case (ii) of 4·11—that the thrust $w\triangle.\frac{2}{3}\beta$ acts at $P$, where $AP = \frac{3}{4}\,AD$, it can be shown at once that this thrust is equivalent to thrust $(w\triangle/3).(\beta/2)$ at $F$ and at $E$ and $(w\triangle/3).\beta$ at $D$.




Thus it has been shown that the theorem is true for every position of a triangle with a vertex in the surface.

Next, let us suppose that the vertex $A$ is lowered to depth $\alpha'$ from the surface. Let the new depths of $B$ and $C$ be $\beta'$ and $\gamma'$ respectively, so that $\beta' = \alpha' + \beta$ and $\gamma' = \alpha' + \gamma$.




On account of the lowering of the triangle through a depth $\alpha'$, there will be an additional thrust

$= w.\, \triangle.\, \alpha'$ acting at the C. G. of $ABC$

$= (w\triangle/3).\alpha'$ at $D + (w\triangle/3).\alpha'$ at $E + (w\triangle/3).\alpha'$ at $F$.

Hence, finally the thrust on the triangle $ABC$

$$= \frac{w\triangle}{3}\left(\frac{\beta+\gamma}{2}+\alpha'\right), \text{ i.e., } \frac{w\triangle}{3}\;\frac{\beta'+\gamma'}{2} \text{ at } D$$

$$+ \frac{w\triangle}{3}\left(\frac{\gamma}{2}+\alpha'\right), \text{ i.e., } \frac{w\triangle}{3}\,.\,\frac{\gamma'+\alpha'}{2} \text{ at } E$$

$$+ \frac{w\triangle}{3}\left(\frac{\beta}{2}+\alpha'\right), \text{ i.e., } \frac{w\triangle}{3}\,.\,\frac{\alpha'+\beta'}{2} \text{ at } F.$$

Thus it is completely proved that *the centre of pressure of any triangle, wholly immersed in a liquid, coincides with the centre of three parallel forces acting at the middle points of the sides and of magnitudes proportional respectively to the depths of their middle points.*

**4·75. Another Proof of the Proposition of 4·7.** From the theorem proved above, the depth of the C. P. of a triangle with its three vertices at depths $\alpha$, $\beta$, $\gamma$, can be obtained. For, since the C. P. of the triangle coincides with the centre of three parallel forces acting at their middle points and proportional in magnitudes to their depths, by taking moments about the line of intersection of the plane of the triangle with the surface we get the

depth of the C. P.

$$=\frac{\frac{w\Delta}{3}\left(\frac{\beta+\gamma}{2}\right)^2+\frac{w\Delta}{3}\left(\frac{\gamma+\alpha}{2}\right)^2+\frac{w\Delta}{3}\left(\frac{\alpha+\beta}{2}\right)^2}{\frac{w\Delta}{3}\cdot\frac{\beta+\gamma}{2}+\frac{w\Delta}{3}\cdot\frac{\gamma+\alpha}{2}+\frac{w\Delta}{3}\cdot\frac{\alpha+\beta}{2}}$$

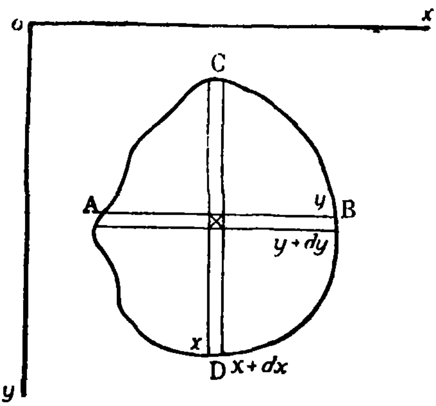
$$=\frac{(\beta+\gamma)^2+(\gamma+\alpha)^2+(\alpha+\beta)^2}{4(\alpha+\beta+\gamma)}$$

$$=\tfrac{1}{2}\cdot\frac{\alpha^2+\beta^2+\gamma^2+\beta\gamma+\gamma\alpha+\alpha\beta}{\alpha+\beta+\gamma}.$$

We see now that if we start with the proposition of 4·7, the result of 4·73 can be deduced from it. Similarly the result of 4·7 can be deduced from that of 4·73.

4·8. **Coordinates of the Centre of Pressure.** Suppose a plane area is immersed in a homogeneous liquid. It is required to find the coordinates of its centre of pressure. For the sake of simplicity we suppose the plane of the area to be vertical.

Let the line of intersection of the plane of the area with the surface of the liquid be taken as the axis of $x$, and a suitable vertical line in the plane of the area as the $y$-axis, the axes being rectangular.




Divide the given area into small elements by lines drawn parallel to the coordinate axes and consider the element $dxdy$ contained in the strips $AB$, $CD$, as shown in the figure.

The thrust on the element $dxdy$ is $wydxdy$, and adding similar expressions for thrusts on all the elements, we see that the total thrust on the given area

$$= \Sigma wydxdy$$
$$= \iint wydxdy$$
$$= w \int (x_B - x_A) ydy \quad \dots \dots \quad (1)$$
$$= w \int AB\, ydy, \quad \dots \dots \dots \quad (2)$$

where $x_B$ and $x_A$ denote the $x$-coordinates of the points $B$ and $A$, and the integration is extended over the whole of the area.

If $x$, $y$ denote the coordinates of the centre of pressure of the given area, then taking moments about the $x$-axis, we get

$$y = \frac{\iint wy^2dxdy}{\iint wydxdy} \quad \dots \quad (3)$$
$$= \frac{\int (x_B - x_A) y^2 dy}{\int (x_B - x_A) y\, dy}$$
$$= \frac{\int ABy^2 dy}{\int ABy\, dy}. \quad \dots \dots \quad (4)$$

Again, taking moments about the $y$-axis, we get

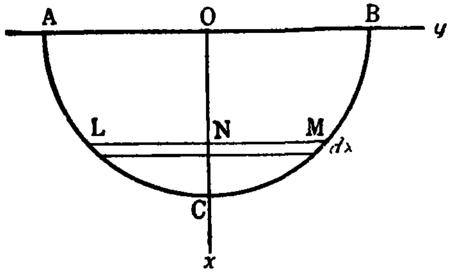
$$x = \frac{\iint wxydxdy}{\iint wydxdy} \quad \dots \dots \quad (5)$$
$$= \frac{1}{2} \frac{\int (x_B{}^2 - x_A{}^2) y\, dy}{\int (x_B - x_A) y\, dy}$$
$$= \frac{1}{2} \frac{\int AB(x_B + x_A) y\, dy}{\int ABy\, dy}. \quad \dots \quad (6)$$

From the knowledge of the equation of the perimeter of the given area, $x_B$, $x_A$ can be determined.

**Note.** It is sometimes found possible to deduce from geometrical consideration, symmetry etc., that the centre of pressure lies on a particular line. In such a case it may no longer be necessary to find both the coordinates of the C. P. by the above formulae, because by finding only one of the coordinates, the other coordinate can be easily inferred. This remark will be illustrated in some of the worked out examples given below.

**4·9. Illustrative Examples.** (1) *A semi-circular lamina is immersed in a liquid with the diameter in the surface. Find the centre of pressure.* [*Agra*, 1928; *Allahabad*, 1939, 1942]

Let the plane of the semi-circle be vertical with the diameter *AOB* in the surface, there being no loss of generality in view of the remark of 4·12.




Let *OC* and *OB* be the two rectangular axes of $x$ and $y$ respectively, and let the equation of the circle be

$$x^2 + y^2 = a^2 \,. \qquad \ldots \quad (1)$$

It is obvious from symmetry that the C. P. will lie on *OC*, so that its $y$-coordinate is zero; hence it is sufficient to find only its $x$-coordinate.

Divide the semi-circle into elementary strips parallel to the y-axis. Let a typical strip *LNM* be of breadth $dx$ at a distance $x$ from the surface.

The area of the strip is $2y\,dx$ and its thrust $wx \cdot 2y\,dx$, where $w$ is the specific weight of the liquid.

If $\bar{x}$ be the $x$-coordinate of the C. P. of the semi-circle, we have

$$\bar{x} = \frac{\int_0^a (2wxy\,dx) \times x}{\int_0^a 2wxy\,dx} = \frac{\int_0^a yx^2\,dx}{\int_0^a yx\,dx}$$

$$= \frac{\int_0^a x^2\sqrt{a^2 - x^2}\,dx}{\int_0^a x\sqrt{a^2 - x^2}\,dx}, \text{ from (1).}$$

The denominator in the above

$$= \Big[-\tfrac{1}{3}(a^2 - x^2)^{3/2}\Big]_a^0 = \tfrac{1}{3}a^3.$$

Putting $\quad x = a\sin\theta$, the numerator

$$= a^4 \int_0^{\pi/2} \sin^2\theta \cos^2\theta\,d\theta$$

$$= a^4 \cdot \tfrac{1}{2} \cdot \tfrac{1}{2}\pi \cdot \tfrac{1}{4} = \tfrac{1}{16}\pi a^4.$$

$$\therefore \quad \bar{x} = \tfrac{3}{16}\pi a.$$

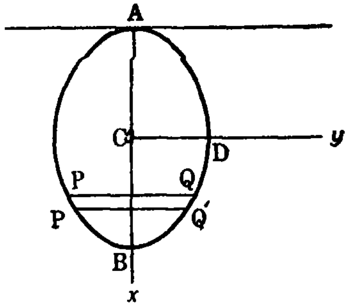
Hence, the C. P. of the semi-circle lies upon the radius perpendicular to the bounding diameter at a distance $\frac{3}{16}\pi a$ from the centre.

(ii) *An ellipse is just immersed in water with its major axis vertical. Show that if the centre of pressure coincides with the focus, the eccentricity of the ellipse must be $\frac{1}{2}$.* [*Agra*, 1926, 1943]

Taking the major and minor axes of lengths $2a$ and $2b$, and as axes of $x$ and $y$ respectively, the equation of the ellipse is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1. \qquad \ldots \ldots (1)$$

It is clear that the C. P. will lie on the line $ACB$.




Taking an elementary strip $PQQ'P'$ parallel to the $y$-axis at depth $x$ below the centre $C$ of the ellipse and of thickness $dx$, the thrust on this strip

$= w(a+x)(2ydx) = 2wy(a+x)dx$,

where $w$ is the specific weight of the liquid.

Now if $Z$ denotes the depth of the C. P. from $C$, we have

$$Z=\frac{\int_{-a}^{a} 2wy(a+x)\times x\,dx}{\int_{-a}^{a} 2wy(a+x)\,dx}. \qquad \ldots (2)$$

The denominator in the above, being the whole pressure on the ellipse,

$= w\,\pi a^2 b$, from **2·81.**

Hence putting the value of $y$ from (1) in the numerator of (2), we have

$$\pi a^2 b.Z = 2b\int_{-a}^{a}(a+x)x\sqrt{\left(1-\frac{x^2}{a^2}\right)}dx.$$

Now putting $x = a\sin\theta$, we get

$$\begin{aligned}
\pi a^2 b.Z &= 2ba^3\int_{-\pi/2}^{\pi/2}(1+\sin\theta)\cos^2\theta\sin\theta\,d\theta\\
&= 2a^3b\int_{-\pi/2}^{\pi/2}(\sin\theta\cos^2\theta+\sin^2\theta\cos^2\theta)\,d\theta\\
&= 2a^3b\int_{-\pi/2}^{\pi/2}\sin^2\theta\cos^2\theta\,d\theta\\
&= 4a^3b\int_{0}^{\pi/2}(\sin^2\theta-\sin^4\theta)\,d\theta\\
&= 4a^3b\left(\tfrac{1}{2}\cdot\pi/2-\tfrac{3}{4}\cdot\tfrac{1}{2}\cdot\pi/2\right).\\
&= 4a^3b\cdot\tfrac{1}{2}(\pi/2)\tfrac{1}{4}=\tfrac{1}{4}\pi a^3 b
\end{aligned}$$

$\therefore \qquad \bar{Z} = \frac{1}{4}a.$

Now the C. P. will coincide with the focus, if

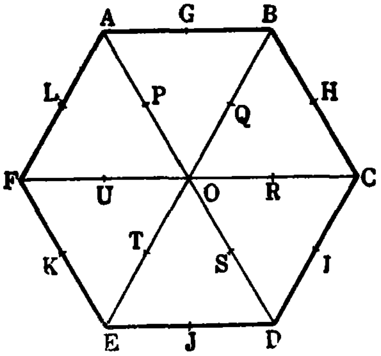
$$\tfrac{1}{4}a = ae,$$

or

$$e = \tfrac{1}{4}.$$

(iii) *A rectangular hexagon of side a is immersed in water with one side in the surface. Show that the depth of its centre of pressure is to that of of its centre of gravity as* 23 : 18. [*U. P. C. S.*, 1939].

Let $ABCDEF$ be the regular hexagon of side $a$, with $AB$ in the




surface. Let $O$ be the centre, and $P, Q, R, S, T, U$; $G, H, I, J, K, L$ be the middle points of $OA, OB, \ldots$ and $AB, CD, \ldots$ respectively.

It is clear that

$$OA = OB = OC = OD = OE = OF = a.$$

The depths below $AB$ of $L, P, Q, H$ is each $\frac{1}{4}\sqrt{3}\,a$.

" " " $F, U, O, R, C$ " $\frac{1}{2}\sqrt{3}\,a$.

" " " $K, T, S, I$ " $\frac{3}{4}\sqrt{3}\,a$.

" " " $E, J, D$ " $\sqrt{3}\,a$.

If the area of each of the six triangles into which the hexagon is divided be $\Delta$, the pressure on each triangle will be equivalent to three parallel forces acting at the mid-points of their sides and equal in magnitude to $\frac{1}{3}w\Delta$ multiplied by their depths.

Hence the pressure on the hexagon will be equivalent to a set of the following forces :—

6 forces each $\dfrac{w\Delta}{3} \cdot \dfrac{\sqrt{3}}{4}a$, acting at a depth $\dfrac{\sqrt{3}}{4}a$,

4 forces each $\dfrac{w\Delta}{3} \cdot \dfrac{\sqrt{3}}{2}a$, acting at a depth $\dfrac{\sqrt{3}}{2}a$,

6 forces each $\dfrac{w\Delta}{3} \cdot \dfrac{3\sqrt{3}}{4}a$, acting at a depth $\dfrac{3\sqrt{3}}{4}a$,

1 force $\dfrac{w\Delta}{3} \cdot \sqrt{3}\,a$, acting at a depth $\sqrt{3}\,a$.

Therefore the depth of the C. P. of the hexagon below $AB$

$$=\frac{\left[\left(6\times\frac{w\Delta}{3}\cdot\frac{3a^2}{16}\right)+\left(4\times\frac{w\Delta}{3}\cdot\frac{3a^2}{4}\right)+\left(6\times\frac{w\Delta}{3}\cdot\frac{27a^2}{16}\right)+\frac{w\Delta}{3}\cdot 3a^2\right]}{\left[\left(6\times\frac{w\Delta}{3}\times\frac{\sqrt{3}a}{4}\right)+\left(4\times\frac{w\Delta}{3}\cdot\frac{\sqrt{3}a}{2}\right)+\left(6\times\frac{w\Delta}{3}\cdot\frac{3\sqrt{3}a}{4}\right)+\frac{w\Delta}{3}\cdot\sqrt{3}a\right]}$$

$$=\sqrt{3}a\left\{\frac{\frac{3}{8}+1+\frac{27}{8}+1}{\frac{3}{2}+2+\frac{9}{2}+1}\right\}$$

$$=\sqrt{3}a\left\{\frac{46}{8}\times\frac{1}{9}\right\}=\frac{23\sqrt{3}}{36}a.$$

The depth of the centre of gravity of the hexagon

$$=\frac{\sqrt{3}}{2}a.$$

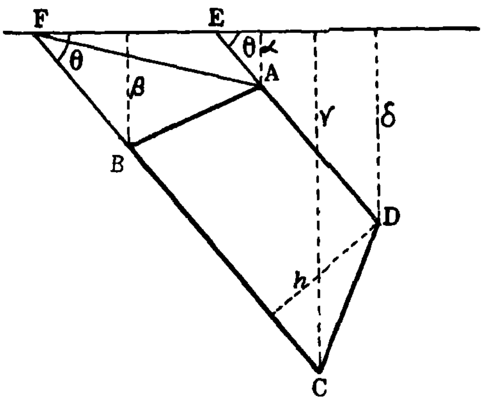
Hence the ratio of the depth of the C. P. to the depth of the C. G.

$$=\frac{23\sqrt{3}}{36}a : \frac{\sqrt{3}}{2}a$$

$$= 23 : 18.$$

(iv) *A trapezium $ABCD$, having $AD$, $BC$ as its parallel sides, is immersed in a liquid with its plane vertical. If the corners $A$, $B$, $C$, $D$ are at depths $\alpha$, $\beta$, $\gamma$, $\delta$ respectively, find the depth of the centre of pressure of the trapezium.*

Let the parallel sides $DA$ and $CB$ be produced to meet the surface of the liquid in $E$ and $F$. Then the given trapezium $ABCD$ may be considered to be the difference of the two trapeziums $EDCF$ and $EABF$.




Let $\theta$ be the inclination of either of the parallel sides to the horizontal and $h$ the perpendicular distance between them.

Now $EA$ being $\alpha$ cosec $\theta$, the area of the $\triangle AEF$ is $\frac{1}{2}h\alpha$ cosec $\theta$ and the thrust on it is $w.\frac{1}{2}h\alpha$ cosec $\theta.\alpha/3$ acting at depth $\frac{1}{2}\alpha$.

Similarly, the thrust on the triangle $ABF$ is

$$w.\tfrac{1}{2}\,h\,\beta\,\mathrm{cosec}\,\theta.\ \ \tfrac{1}{3}(\alpha+\beta)$$

acting, by the formula of 4·71, at a depth

$$\tfrac{1}{2}\frac{\alpha^2+\alpha\beta+\beta^2}{\alpha+\beta}.$$

Hence the thrust on the trapezium $EABF$ is

$$\tfrac{1}{6}\,wb\,\mathrm{cosec}\,\theta\,(\alpha^2+\alpha\beta+\beta^2),$$

and it acts at a depth

$$=\frac{\tfrac{1}{6}\,wb\,\mathrm{cosec}\,\theta\,.\,\alpha^2\times\tfrac{1}{2}\,\alpha+\tfrac{1}{6}\,wb\,\mathrm{cosec}\,\theta\,.\,\beta(\alpha+\beta)\times\dfrac{(\alpha^2+\alpha\beta+\beta^2)}{2(\alpha+\beta)}}{\tfrac{1}{6}\,wb\,\mathrm{cosec}\,\theta\,(\alpha^2+\alpha\beta+\beta^2)}$$

$$=\tfrac{1}{2}\frac{\alpha^3+\beta(\alpha^2+\alpha\beta+\beta^2)}{\alpha^2+\alpha\beta+\beta^2},$$

$$=\tfrac{1}{2}\frac{(\alpha+\beta)(\alpha^2+\beta^2)}{\alpha^2+\alpha\beta+\beta^2}.$$

Similarly, the thrust on the trapezium $EDCF$ is

$$\tfrac{1}{6}\,wb\,\mathrm{cosec}\,\theta\,(\gamma^2+\gamma\delta+\delta^2),$$

acting at depth

$$\tfrac{1}{2}\frac{(\gamma+\delta)(\gamma^2+\delta^2)}{\gamma^2+\gamma\delta+\delta^2}.$$

Hence the depth of the C. P. of the given trapezium

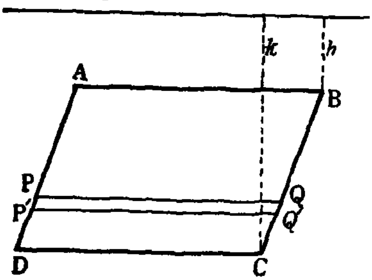
$$=\frac{\left[\tfrac{1}{6}\,wb\mathrm{cosec}\theta\,(\gamma^2+\gamma\delta+\delta^2)\times\tfrac{1}{2}\dfrac{(\gamma+\delta)(\gamma^2+\delta^2)}{\gamma^2+\gamma\delta+\delta^2}-\tfrac{1}{6}\,wb\,\mathrm{cosec}\,\theta(\alpha^2+\alpha\beta+\beta^2)\,.\,\tfrac{1}{2}\dfrac{(\alpha+\beta)(\alpha^2+\beta^2)}{\alpha^2+\alpha\beta+\beta^2}\right]}{\tfrac{1}{6}\,wb\,\mathrm{cosec}\,\theta\,(\gamma^2+\gamma\delta+\delta^2)-\tfrac{1}{6}\,wb\,\mathrm{cosec}\,\theta\,(\alpha^2+\alpha\beta+\beta^2)}$$

$$=\tfrac{1}{2}\frac{(\gamma+\delta)(\gamma^2+\delta^2)-(\alpha+\beta)(\alpha^2+\beta^2)}{(\gamma^2+\gamma\delta+\delta^2)-(\alpha^2+\alpha\beta+\beta^2)}.$$

(v) *Prove that the depth of the C. P. of a parallelogram two of whose sides are horizontal and at depths h, k below the surface of a liquid whose density varies as the depth below the surface, is*

$$\tfrac{3}{4}\frac{h^3+h^2k+hk^2+k^3}{h^2+hk+k^2}.$$ [*Calcutta*, 1914]

Let $PQQ'P'$ be an elementary strip of area of small breadth $dz$, parallel to the horizontal side of the parallelogram, and let its depth below the surface be $z$.




Since the density varies as $z$, the pressure will vary as $z^2$ (vide question 11, Ex. II). Let the pressure be $\mu z^2$, where $\mu$ is a constant.

The pressure on the elementary strip of the area

$$= (\mu z^2)(PQ\,dz).$$

Now, if $\bar{z}$ denotes the depth of the C. P. of the parallelogram, we have

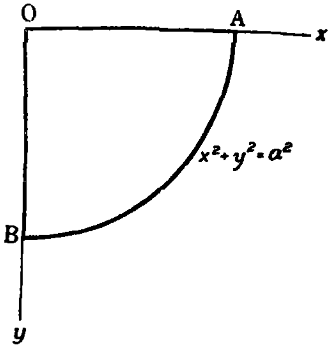
$$\bar{z} = \frac{\int_h^k (\mu z^2)(PQ\,dz)z}{\int_h^k (\mu z^2)(PQ\,dz)}$$

$$= \frac{\int_h^k z^3 dz}{\int_h^k z^2 dz}$$

$$= \frac{3}{4}\frac{(k^4 - h^4)}{(k^3 - h^3)}$$

$$= \frac{3}{4}\frac{h^3 + h^2k + hk^2 + k^3}{h^2 + hk + k^2}.$$

(vi) *A quadrant of a circle is just immersed vertically in a heavy homogeneous liquid with one edge in the surface. Find the centre of pressure.*




Let the two edges $OA$, $OB$ be the rectangular axes of $x$ and $y$ respectively. If $a$ be the radius, the equation of the curve is

$$x^2 + y^2 = a^2. \quad . \quad . \quad . \quad . \quad (1)$$

Denoting by $\bar{x}$ and $\bar{y}$ the $x$ and $y$ coordinates of the C. P. of the quadrant, we get by making use of the formulae (3) and (5) of 4·8

$$\bar{x} = \frac{\iint xy\,dx\,dy}{\iint y\,dx\,dy} = \frac{\int_0^a xy^2\,dx}{\int_0^a y^2\,dx}$$

$$= \frac{\int_0^a x(a^2 - x^2)\,dx}{\int_0^a (a^2 - x^2)\,dx}, \text{ from (1)}$$

$$= \frac{\left[\frac{1}{2}a^2x^2 - \frac{1}{4}x^4\right]_0^a}{\left[a^2x - \frac{1}{3}x^3\right]_0^a} = \frac{\frac{1}{4}a^4}{\frac{2}{3}a^3}$$

$$= \frac{3}{8}a.$$

Again,

$$\bar{y} = \frac{\iint y^2\,dx\,dy}{\iint y\,dx\,dy} = \frac{2}{3}\frac{\int_0^a y^3\,dx}{\int_0^a y^2\,dx}$$

$$= \frac{2}{3}\frac{\int_0^a (a^2 - x^2)^{3/2}dx}{\int_0^a (a^2 - x^2)\,dx}, \text{ from (1).}$$

Now, putting $x = a \sin\theta$, we have

$$\int_0^a (a^2 - x^2)^{3/2}\,dx$$

$$= a^4\int_0^{\pi/2} \cos^4\theta\,d\theta = a^4 \cdot \tfrac{3}{4} \cdot \tfrac{1}{2} \cdot \pi/2.$$

$$= \tfrac{3}{16}\pi a^4.$$

Also
$$\int_0^a (a^2 - x^2)\,dx = \tfrac{2}{3}a^3.$$

Hence

$$\bar{y} = \tfrac{2}{3}\left[\tfrac{3}{16}\pi a^4 \div \tfrac{2}{3}a^3\right] = \tfrac{3}{16}\pi a.$$

Thus the C. P. is $(\frac{3}{8}a, \frac{3}{16}\pi a)$.

**Note.** On comparing the above example with the example (1) we observe that the depth of the C. P. below the surface of the liquid would be the same in both cases. But in ex. (1) it was evident from symmetry that the C. P. would lie on the vertical radius, so that in order to determine the position of C. P., it was quite sufficient to find only its depth. In ex. (vi), however, it becomes necessary to find both the coordinates in order to locate the situation of the centre of pressure.

## Examples IX

1. One end of a horizontal pipe of circular section is closed by a vertical door hinged to the pipe at the top. Find the horizontal force that must be applied through the centre of the door on the other side of it, so as to keep it just closed, the pipe being just full of a heavy liquid.

2. A hole in the side of a ship is closed by a circular door 5 ft. in diameter hinged at the highest point and held inside against the water pressure by a fastening at its lowest point. If the highest and the lowest points of the door are at depths of 4 and 8 ft., show that the least force exerted by the fastening must be 1·78 tons.

[*M. T.; U. P. C. S.*, 1942]

3. The vertical wall of a water tank contains a circular door whose diameter is 2 ft. and whose centre is 3 ft. below the surface of water, the door being hinged at its lowest point. Show that the force on the fastening is about 319 lbs. wt. [*Lucknow*, 1941]

4. Show that the centre of pressure of a triangular lamina, the depths of whose angular points are $\alpha$, $\beta$, $\gamma$, is at a depth

$$\frac{1}{4} \cdot \frac{\alpha^2 + \beta^2 + \gamma^2 - \beta\gamma - \gamma\alpha - \alpha\beta}{\alpha + \beta + \gamma}$$

below the centre of gravity of the lamina.

5. Show that the depth of the centre of pressure of a rhombus totally immersed with one diagonal vertical and its centre at a depth $h$, is $h + a^2/(24h)$, where $a$ is the length of the vertical diagonal.

[*Allahabad*, 1924, 1934]

6. A parallelogram has its corners at depths $h_1$, $h_2$, $h_3$, $h_4$ below the surface of a liquid and its centre at a depth $h$, show that the depth of its centre of pressure is

$$(h_1^2 + h_2^2 + h_3^2 + h_4^2 + 8h^2)/(12h).$$

[*Agra*, 1941, *Allahabad*, 1942]

7. A parallelogram whose plane is vertical and centre at a depth $h$ below the surface, is totally immersed. Show that if $a$ and $b$ are the lengths of the projections of its sides on a vertical line, then the depth of its centre of pressure will be

$$h + \frac{a^2 + b^2}{12h}$$

8. A square whose side is $2a$ is completely immersed in a homogeneous liquid in a vertical plane with its centre at a depth $d$. Prove that the centre of pressure is vertically below the centre of the square and at a distance $\frac{1}{3}a^2/d$ from it, whatever be the inclination of the sides of the square to the vertical. [*M. T.*]

9. A uniform elliptic lamina whose axes are $2a$ and $2b$, is half immersed in water, the axis $2b$ being in the surface. Find the centre of pressure. [*Lucknow*, 1942]

10. An ellipse is completely immersed, with its minor axis horizontal and at a depth $h$; find the position of the centre of pressure. [*Allahabad*, 1943]

11. A segment of a parabola, cut off by a double ordinate at a distance $h$ from the vertex, is immersed with this ordinate in the surface of a liquid, find the resultant thrust and the position of the centre of pressure on the lamina, neglecting the atmospheric pressure.

[*Allahabad*, 1937]

12. A semi-ellipse bounded by its minor axis is just immersed in a liquid, the density of which varies as the depth; if the minor axis be in the surface, find the eccentricity in order that the focus may be the centre of pressure. [*Allahabad*, 1933; *Agra*, 1937]

13. A circular area is immersed in a liquid with its plane vertical and its centre at a depth $c$. If the area is lowered slowly so that the centre is at a depth $f(t)$ at time $t$, show that the velocity of the centre of pressure at that instant is

$$f'\{1 - a^2/(4f^2)\},$$

where $a$ is the radius. [*Bombay*, 1935]

14. A flat circular plate of radius $a$, lies on a plane inclined at $30^\circ$ to the horizontal, and is subjected to water pressure on one face. The centre of pressure is at a distance $a/16$ from the geometric centre. Show that the geometric centre is at a depth $2a$ below the free surface of the water. [*M T*]

15. A regular hexagon is immersed in water with one side in the surface; find the distance between the centres of pressure of the two halves into which it is divided by the horizontal diagonal.

16 A square lamina $ABCD$ which is immersed in water, has the side $AB$ in the surface, draw a line $BE$ to a point $E$ in $CD$ such that the pressures on the two portions may be equal Prove that, if this be the case, the distance between the centres of pressure of the two portions is $(\sqrt{505}/48)a$, where $a$ is the side of the square

17. A rectangle is immersed in two liquids of density $\rho$ and $2\rho$ which do not mix; the top of the rectangle is in the surface of the first liquid, and the area immersed in each is the same, prove that the centre of pressure divides the rectangle in the ratio 7 3

18. Prove that the centre of pressure of a rhombus immersed in two liquids which do not mix, with a vertex in the upper surface and a diagonal in the common surface, divides the other diagonal in the ratio of 17 : 11, if the density of the lower liquid is twice that of the upper liquid.

19. A fluid of depth $2a$ and uniform density $\rho$ is superposed on a liquid of density $2\rho$ and depth greater than $a$ A circular lamina of radius $a$, is placed with its plane vertical and its centre in the surface common to the two liquids. Determine the depth of the centre of pressure below the centre of the lamina, neglecting the atmospheric pressure. [*M. T.*]

20. If a quadrilateral area is entirely immersed in water, and $\alpha, \beta, \gamma, \delta$ be the depths of its four corners, and $h$ the depth of its centre of gravity, show that the depth of its centre of pressure is
$\frac{1}{2}(\alpha + \beta + \gamma + \delta) - \frac{1}{6}(\beta\gamma + \gamma\alpha + \alpha\beta + \alpha\delta + \beta\delta + \gamma\delta)/h.$
[*M. T.*]

21. A rectangle is immersed in fluids of density $\rho, 2\rho, 3\rho, \ldots, n\rho$; the top of the rectangle being in the surface of the first liquid and the area immersed in each liquid being the same, show that the depth of the centre of pressure of the rectangle is

$$\frac{3n+1}{2n+1}\,h/2,$$

where $h$ is the depth of the lower side.

22. An area bounded by the curve $ay^2 = x^3$, the $x$-axis and the ordinate $x = a$, is immersed in water with the axis in the surface. Find the coordinates of the centre of pressure.

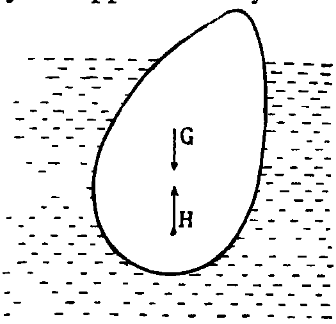
23. Find the coordinates of the centre of pressure of the area between the curve $\sqrt{x} + \sqrt{y} = \sqrt{a}$, and the axes, taking the axes to be rectangular and one of them in the surface.

CHAPTER V

# EQUILIBRIUM OF FLOATING BODIES

**5·1. General considerations for equilibrium.** When a solid body is floating in a liquid at rest, then besides the gravitational force the body may or may not be subject to other external forces. For instance, it may be floating freely or floating under constraint in the sense that one or two points of it may be fixed, or it may be subject to a tensional force applied to it by means of a string attached to some point of it, and so on. But whatever be the nature of the external forces, there is always present one force associated with a floating body and it is the upward vertical thrust or the force of buoyancy which in accordance with Archimedes' principle, is equal to the weight of the liquid displaced by the body. Therefore, whenever we have to deal with the equilibrium of a floating body, we have to take into consideration the weight of the body, the force of buoyancy and other external forces, if any. We then apply to these the principles which are used in *Statics* to treat the equilibrium of a set of forces acting on a body.

**5·2. Bodies floating freely.** Suppose a body floats freely, wholly or partially immersed, in a liquid which is at rest under gravity.




The vertical forces acting on the body are

(1) its weight acting downwards through its centre of gravity;

(2) the force of buoyancy

which is equal to the weight of the liquid displaced and acts vertically upwards through the centre of buoyancy, which is the C. G. of the displaced liquid.

The necessary and sufficient conditions for equilibrium are that these two forces should be equal and opposite and act along the same straight line.

Hence the conditions of equilibrium for a body floating freely in a liquid are:

(i) *The volume of the liquid displaced must be such that its weight would be equal to the weight of the body.*

(ii) *The position of the body should be such that the centres of gravity of the body and of the displaced liquid must be in the same vertical line.*

**5·21. Volume immersed.** If a *solid of volume $V$ and mean density $\varrho$ floats in a liquid of density $\varrho'(>\varrho)$, the volume immersed is $V.\varrho/\varrho'$.*

Suppose $V'$ is the volume of the solid immersed. Then since the weight of the liquid displaced is equal to the weight of the solid, we have

$$g\varrho' V' = g\varrho V,$$

or

$$V' = V.\varrho/\varrho'.$$

If $\varrho > \varrho'$, the solid obviously cannot float.

**Note.** In the above proposition we have used the term ***mean*** density which may be explained thus. Suppose a hollow ball of volume $V$ and weight $W$ is made out of a metal, say iron. Then $\varrho$ will be called the ***mean density*** of the ball, if $g\varrho V = W$. Clearly $\varrho$ will depend upon the volume $V$ of the ball and will differ from the actual density of iron. In calculating ***mean density*** of the ball, therefore, we must take into account also the space inside.

Thus we see that although the density of iron is greater than that of water, a piece of iron may be made into a shape so that its ***mean density*** would become less than that of water in which it may then float.

**5·22. Bodies floating in more than one liquid.** Suppose a body floats partly immersed in one liquid and partly in another. Then for equilibrium, *the sum of the weights of the displaced portions of the two liquids must be equal to the weight of the body and the centres of gravity of the body and of the whole liquid displaced must be in the same vertical line.*

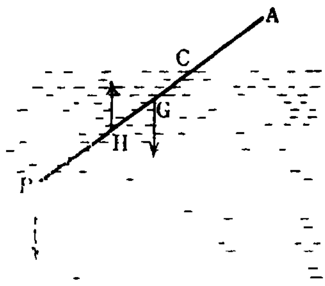
This includes the case of a body which floats with part of its volume immersed in a liquid and the rest in air.

It is clear that a similar result will be true if the body be floating immersed in parts in more than two liquids.

**5·3. Illustrative Examples.** (1) *A rod of small section and density $m$, has a small piece of lead of weight $1/n$ th that of the rod attached to one extremity; prove that the rod will float at any inclination in a liquid of density $m'$, if*

$$(n+1)^2 m = n^2 m'. \qquad \text{[Lucknow, 1936, 1942; Agra, 1929, 1935]}$$

Let $AB$ be the rod floating at any inclination with its part $CB$ immersed in the liquid. Then $G$ and $H$ being the middle points of $AB$ and $CB$ respectively are their centres of gravity.




Suppose $AB$ and $CB$ are of lengths $2a$ and $2x$ respectively and $k$ is the uniform corss-section of the rod.

The vertical forces acting on the rod are:

(i) The weight $2akmg$ of the rod acting downwards through $G$.

(ii) The weight $2akmg/n$ of the lead acting downwards at $B$.

(iii) The force of buoyancy $2xkm'g$ equal to the weight of the liquid displaced, acting upwards at $H$.

For equilibrium, therefore, we have

$$2xkm'g = 2akmg + 2akmg/n,$$

or
$$xm' = am + \frac{am}{n},$$

or
$$x = \frac{am}{m'}\left(\frac{n+1}{n}\right). \qquad \ldots (1)$$

Now taking moments about $B$, we get

$$2xkm'g \times x = 2akmg \times a,$$

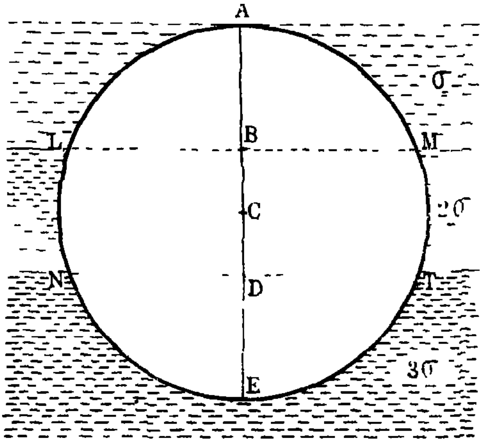
or
$$x^2m' = a^2m,$$

or
$$\frac{a^2m^2}{m'^2}\left(\frac{n+1}{n}\right)^2 m' = a^2m, \quad \text{from (1)},$$

or
$$(n+1)^2m = n^2m'.$$

(ii) *A solid sphere floats just immersed in a heterogeneous liquid composed of three liquids which do not mix and whose densities are as* 1 : 2 : 3. *If the thickness of the two upper liquids be each one-third of the diameter of the sphere, show that the density of the liquid in the middle is equal to the density of the sphere.* [*Allahabad*, 1938]

Let $C$ be the centre of the circle of radius $a$ and $ACE$ its vertical




diameter. Let $\rho$ be the density of the sphere and $\sigma$, $2\sigma$ and $3\sigma$ those of the three liquids.

There are four vertical forces acting; three are the upward forces of buoyancy each equal to the weight of the respective portion of one of the three liquids displaced, and the fourth is the weight of the sphere acting downwards.

The volumes of the uppermost and lowest liquids displaced are equal and each

$$= \tfrac{1}{3}\pi \cdot (2a/3)\,[3a^2 - (a^2 + a^2/3 + a^2/9)]$$
$$= 2\pi a^3/9 \times 14/9 = 28\pi a^3/81 .$$

The volume of the middle liquid displaced

$$= \tfrac{1}{3}\pi \cdot (2a/3) \cdot [3a^2 - (a^2/9 - a^2/9 + a^2/9)]$$
$$= \tfrac{52}{81}\pi a^3.$$

Hence the weights of the three displaced liquids are respectively

$$\tfrac{28}{81}\pi a^3\sigma g,\ \tfrac{52}{81}\pi a^3 .\ 2\sigma g,\ \tfrac{28}{81}\pi a^3 .\ 3\sigma g.$$

Since all the four forces pass through the same vertical line $ACE$, we have

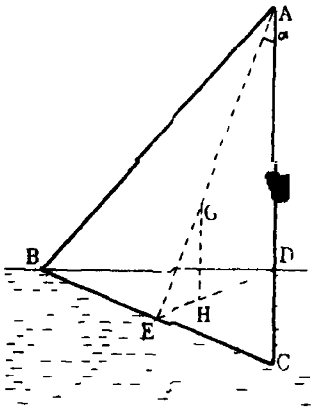
$$\tfrac{28}{81}\pi a^3\sigma g + \tfrac{104}{81}\pi a^3\sigma g + \tfrac{84}{81}\pi a^3\sigma g = \tfrac{4}{3}\pi a^3\varrho g,$$

or $$\tfrac{216}{81}\sigma = \tfrac{4}{3}\varrho,$$

or $$\varrho = 2\sigma.$$

(iii) *A uniform prism, whose cross-section is an isosceles triangle of vertical angle* $2\alpha$, *floats freely in a liquid with its base just immersed, one edge being in the surface; show that the ratio of its density to that of the liquid is* $2\sin^2\alpha$. [*M. T.*]

Let $ABC$ be the section through the C. G. of the prism. Let the immersed portion be $BCD$, so that $BD$ is horizontal. Let $G$ and $H$ be the centres of gravity and buoyancy respectively, $E$ the middle point of $BC$ and $\angle A = 2\alpha$.




The conditions of equilibrium are that

(1) the line $GH$ must be vertical, and

(2) the weight of the prism must be equal to the weight of the liquid displaced.

Since $EG = \frac{1}{3}AE$ and $EH = \frac{1}{3}ED$, $GH$ is parallel to $AD$. But $GH$ being vertical from (1), $AC$ also must be vertical.

Now, if $l$ be the length of the prism, $\varrho$ and $\sigma$ the densities of the prism and the liquid respectively, we get from (2)

$$g\varrho \times \text{area } ABC \times l = g\sigma \times \text{area } BCD \times l,$$

or
$$\frac{\varrho}{\sigma} = \frac{\triangle BCD}{\triangle ABC} = \frac{\frac{1}{2}BD \times CD}{\frac{1}{2}BD \times AC}$$
$$= \frac{CD}{AC} = \frac{BD\tan\alpha}{AC}$$
$$= \frac{AB\sin 2\alpha\tan\alpha}{AC}$$
$$= \frac{AC\,2\sin\alpha\cos\alpha\tan\alpha}{AC}$$
$$= \sin^2\alpha.$$

## Examples X

1. What fraction of a piece of cork will be outside water in which it floats, the specific gravity of cork being 0·24?

2. The specific gravity of sea-water is 1·028 and of ice 0·918. What fraction of the volume of an iceberg floats out of water?

3. A piece of wood weighing 24 grammes floats in water with $\frac{2}{3}$rd of its volume immersed. Find the density and volume of the wood.

4. The specific gravity of ice is 0·92, that of the sea-water is 1·025. What depth of water will be required to float a cubical iceberg whose side is 100 feet?

5. How much water will overflow from the edges of a cup just full of water when a cork 2 cubic inches in volume is gently placed in it so as to float? (Take the sp. gr. of cork 0·24.)

6. A vessel contains water and mercury. A cube of iron 5 cm. along each edge, is in equilibrium in the liquids, with its faces vertical and horizontal. Determine how much it is in each liquid, the densities of iron and mercury being 7·7 and 13·6 respectively.

7. A uniform cylinder when floating with its axis vertical in distilled water, sinks to a depth of 3·2″, and when floating in alcohol sinks to a depth of 4″. Find the specific gravity of alcohol.

8. A wooden solid cube of side one metre floats in water with $\frac{3}{5}$ths of its volume immersed. Calculate the depth to which it will sink in a liquid of sp. gr. 0·8.

9. A rectangular block of boxwood 10 cm. in depth and of sp. gr. 0·9 is floating in water with its upper surface horizontal. Oil of sp. gr. 0·6 is poured on to the water, so as just to cover the block of wood. Show that neglecting the buoyancy of the air, the wood will rise through 1·5 cm.

10. A vertical cylinder of density $7\rho/4$ floats completely immersed in two liquids, the density of the upper liquid is $\rho$ and that of the lower $2\rho$; find the position of equilibrium.

11. A cylinder of sp. gr. $\sigma$, floats with its axis vertical partly in one fluid of sp. gr. $\sigma_1$ and partly in another of sp. gr. $\sigma_2$. Show that the common surface divides the axis in the ratio of $\sigma-\sigma_2 : \sigma_1-\sigma$.

12. A right circular cone, of density $\rho$, floats just immersed with its vertex downwards in a vessel containing two liquids of densities $\sigma_1$ and $\sigma_2$ respectively; show that the plane of separation of the two

liquids cuts off from the axis of the cone a fraction $\sqrt[3]{(\rho - \sigma_2)/(\sigma_1 - \sigma_2)}$. of its length. [*Allahabad*, 1939]

13. A solid homogeneous right cone of sp. gr. $\sigma$ floats in a given homogeneous liquid; find the position of equilibrium firstly when the vertex is down and the base up; and secondly when the base is down and the vertex up.

14. A hollow conical vessel floats in water with its vertex downwards and a certain depth of its axis immersed; when water is poured into it upto the level originally immersed, it sinks till its mouth is on a level with the surface of the water. What portion of the axis was originally immersed? [*Agra*, 1936, *M. T.*]

15. A frustum of a right circular cone, cut off by a plane bisecting the axis perpendicularly, floats with its smaller end in water and its axis just half immersed. Prove that the sp. gr. of the cone is 19/56.

16. A man whose weight is equal to 160 lbs and whose sp. gr. is 1·1, can just float in water with his head above the surface by the aid of a piece of cork which is wholly immersed. Having given that the volume of his head is one sixteenth of his whole volume and that the sp. gr. of cork is 0·24, find the volume of the cork, the weight of a cubic foot of water being 62·5 lbs. [*Allahabad*, 1931]

17. A closed cubical vessel with walls one inch in thickness is to be made of a metal whose sp. gr. is 27/19 Show that in order that the vessel may float in water its internal volume must be at least 64 cubic inches.

18. A cubical box of one foot external dimension is made of material of thickness one inch and floats in water immersed to a depth of $3\frac{3}{8}$ inches. Not taking air into consideration, determine the weight of the box and the amount of water that must be poured in so that the water inside and outside may stand at the same level. [*Agra*, 1938]

19. A steamer in going from salt water into fresh water was observed to sink 2 inches, but after burning 50 tons of coal to rise one inch. Supposing the densities of salt and fresh water to be as 65 · 64, find the displacement of the steamer in tons. [*Lucknow*, 1934]

20. A ship sailing from the sea into a river sinks $a$ inches, and on discharging $x$ tons of her cargo rises $b$ inches. If sea-water be one-fortieth heavier than river water, prove that the weight of the ship

is 41 $(a/b)\,x$ tons. Assume the sides of the ship to be vertical at water level. [*Benares Eng.*, 1943]

21. A ship sailing out of the sea into a river sinks through a distance $b$ feet; on unloading a cargo of weight $P$, the ship rises through $c$ feet; show that the weight of the ship after unloading is

$$\left(\frac{b\sigma}{c(\sigma-\varrho)}-1\right)\varrho,$$

where $\sigma$ are $\varrho$ are the sp. gr. of sea water and river water respectively. The cross-section of the ship near the level of water may be assumed to be uniform. [*Lucknow*, 1931, 1941]

22. A solid displaces $\frac{1}{2}$, $\frac{1}{3}$ and $\frac{1}{4}$ of its volume respectively when it floats in three different fluids; find the volume it displaces when it floats in a mixture formed, first of equal volumes of the fluids; second, of equal weights of the fluids. [*Agra*, 1935]

23. A piece of wood floats in a vessel of water with $\frac{4}{5}$ths of its volume immersed. Find the additional volume which will be immersed when the vessel is placed in a vacuum, the sp. gr. of air being 0·0013.

24. A cylinder of wood (sp. gr. $\frac{3}{4}$) of length $b$, floats with its axis vertical in water and oil (sp. gr. $\frac{1}{2}$), the length of the solid in contact with the oil being $a\ (<b/2)$. Find how much of the wood is above the liquids; also find to what additional depth must oil be added so as to cover the cylinder. [*I. C. S.*, 1940]

25. A small hole is drilled at one end of a thin uniform rod and it is filled with some much denser metal. It is observed that the rod can float in water half immersed and inclined at any angle to the vertical. Show that the sp. gr. of the rod is $\frac{1}{4}$.

26. *A triangular lamina ABC of density $\varrho$ floats in a liquid of* density $\sigma$ with its plane vertical, the angle $B$ being in the surface of the liquid, and the angle $A$ not immersed. Show that

$$\varrho : \sigma = \sin A \cos C : \sin B. \qquad [\textit{Lucknow}, 1933]$$

**5·4. A body wholly immersed in a liquid and supported by a string.** If a body rests totally immersed in a liquid and is supported by a string, then the vertical forces acting on the body are :

(i) The tension of the string acting upwards when the density of the body is greater than that of the liquid.

(ii) The force of buoyancy which is equal to the weight of the liquid displaced, and acts upwards.

(iii) The weight of the body acting downwards.

Hence, for equilibrium, we have

Tension of the string + wt of the liquid displaced
= wt. of the body.

i.e. *Tension of the string*

= *wt. of the body — wt. of the liquid displaced,* . (1)

or $$T = W - W', \quad . \quad . \quad (2)$$

where $T$, $W$, $W'$ denote respectively the tension, the wt. of the body and the wt. of the liquid displaced.

**5·41. Weighing a body immersed in a liquid.** If a body while immersed in a liquid, is weighed by a balance, then the tension of the string supporting the body will not indicate its true weight; it will indicate, as shown in the previous article, only its *apparent* weight, which is the difference of its real weight and the weight of the liquid displaced by the body.

Hence whenever a body is weighed in a liquid, the weight indicated by the balance is less than the actual weight by the weight of the liquid displaced.

Obviously a similar discrepancy will arise when a body is weighed in air, because the quantities of air displaced by the body and by the "weights" that are used, are generally not the same.

**5·42. Corrections for weighing in air.** In order to ascertain the perfectly accurate weight of a body, the weighing should be done in *vacuo*. But this is neither convenient nor always possible. Hence when greater accuracy is desired, a correction is applied to the apparent weight of the body obtained by weighing it in air. This is done, as explained below, by knowing the densities of

the body, the substance of which the "weights" are made and of the air.

*When a body of density* $\varrho$, *is weighed in air by means of "weights" whose density is* $\varrho'$, *to find the true weight of the body corresponding to an apparent weight* $W_0$, $\sigma$ *being the density of the air.*

Let $W$ be the true weight of the body corresponding to its apparent weight $W_0$, which is the sum of the "weights" required to balance the body on the other pan.

Assuming the balance to be correct, the two forces supporting the scale pans must be equal Hence, we have

wt. of the body — wt. of the air displaced by it
= wt. of the "weights" — wt. of the air displaced by the "weights",

or
$$W - \frac{W}{\varrho g}\cdot\sigma g = W_0 - \frac{W_0}{\varrho' g}\cdot\sigma g, \quad . \quad . \quad . \quad (1)$$

for, from (1) of 1·31, the volume of a substance of weight $W$ and density $\varrho$ is $W/\varrho g$.

Hence, we get

$$W = W_0 \cdot \frac{1 - \sigma/\varrho'}{1 - \sigma/\varrho}. \quad . \quad . \quad . \quad (2)$$

Since in general $\sigma$ is much smaller than $\varrho$ and $\varrho'$, we may use the Binomial theorem and have

$$W = W_0(1 - \sigma/\varrho')(1 - \sigma/\varrho)^{-1}$$
$$= W_0(1 - \sigma/\varrho')(1 + \sigma/\varrho + \sigma^2/\varrho^2 + \ldots). \quad . \quad (3)$$

Hence, neglecting square and higher powers of $\sigma$, we get a fairly approximate value of the true weight $W$ from the equation

$$W = W_0(1 - \sigma/\varrho' + \sigma/\varrho). \quad . \quad . \quad (4)$$

**5·5. Illustrative Examples.** (i) *If* $W$ *and* $w$ *be the weights of a body in vacuo and water respectively, prove that the weight in air of sp. gr.* $\sigma$ *will be* $W - \sigma(W - w)$. [*Patna*, 1941]

The true weight of the body being $W$, its weight in water is

$$w = W - \text{wt. of the displaced water}$$
$$= W - vw',$$

where $v$ is the volume of the body and $w'$ the weight of unit volume of water.

$$\therefore \quad vw' = W - w. \qquad \ldots \quad (1)$$

Now the wt. of the body in air

$$= W - \text{wt. of the air displaced}$$
$$= W - v\sigma w'$$
$$= W - \sigma(W - w), \quad \text{from (1)}.$$

(ii) *A piece of metal weighing 32 lbs. in water is attached to a piece of wood whose weight is 30 lbs. and then the compound body is found to weigh 12 lbs. in water. Prove that the sp. gr of the wood is 0·6*

Suppose the volumes of the metal and the wood are $v_1$ and $v_2$ respectively and $\sigma_1$ and $\sigma_2$ their sp. gr. If $w$ denotes the weight of unit volume of water, we have

$$v_2\sigma_2 w = 30, \qquad \ldots \quad (1)$$
$$v_1\sigma_1 w - v_1 w = 32, \qquad \ldots \quad (2)$$
$$v_1\sigma_1 w + v_2\sigma_2 w - (v_1 + v_2) = 12. \qquad \ldots \quad (3)$$

By substituting from (1) and (2) in (3), we get

$$30 + 32 - v_2 w = 12,$$

or

$$v_2 w = 50.$$

$$\therefore \quad \sigma_2 = 30/50 = 0 \cdot 6$$

## Examples XI

1. A piece of silver and a piece of copper fastened to the ends of a string passing over a pulley, hang in equilibrium when suspended in a liquid of density 1 15. Determine the relative volumes of the masses, the densities of silver and copper being 10 47 and 8 89 respectively.

2. A piece of silver and a piece of gold are suspended from the ends of a balance beam which is in equilibrium when the silver is immersed in alcohol and the gold in nitric acid, the densities of gold, silver, nitric acid and alcohol being 19 3, 10 5, 1 5 and 0·85 respectively. Compare the masses of gold and silver.

3. The apparent weight of a piece of platinum when immersed in water is 20·6 gr., when immersed in mercury it is 8 gr. The density

of mercury being 13·6, find the volume and density of the platinum.

4. 37 lbs. of tin loses 5 lbs. in water, 23 lbs. of lead loses 2 lbs. in water; a composition of lead and tin weighing 120 lbs. loses 14 lbs. in water. Find the proportion of lead to tin in the composition.

5. A body *A* weighing 3 grammes is attached to another body *B* weighing 6 grammes, and the whole immersed under water, when they are found to weigh 2 grammes. The body *B* under water alone weighs 4 grammes. Find the sp. gravities of *A* and *B*.

6. The apparent weight of a sinker weighed in water, is four times the weight in vacuum of a piece of material, whose sp. gr. is required; that of the sinker and the piece together is three times that weight. Show that the sp. gr. of the material is 0·5. [*M. T.*]

7. If a body of mass $m_1$ and density $\zeta_1$ is balanced by a mass $m_2$ and density $\zeta_2$ when placed on the pans of a common balance, show that

$$m_1 = m_2 \frac{\zeta_1}{\zeta_2}\left(\frac{\zeta_2 - \zeta_3}{\zeta_1 - \zeta_3}\right),$$

where $\zeta_3$ is the density of the air. [*Benares*, 1940]

8. A body consists of an alloy of two metals of sp. gravities $s_1$ and $s_2$ respectively; its weight in vacuo is $w$ and in water is $w'$. Show that the proportion of the two metals by volume is

$$s_2 w' - (s_2 - 1)\, w : (s_1 - 1)\, w - s_1 w'.$$

9. A body floats in a fluid of sp. gr. $s$ with as much of its volume out of the fluid as would be immersed in a second fluid of sp. gr. $s'$, if it floated in that fluid. Prove that the sp. gr. of the body is $ss'/(s+s')$.

10. Two solids are each weighed in succession in three homogeneous liquids of different densities; if the weights of the one are $w_1$, $w_2$ and $w_3$ and those of the other are $W_1$, $W_2$ and $W_3$, prove that

$$w_1 (W_2 - W_3) + w_2 (W_3 - W_1) + w_3 (W_1 - W_2) = 0.$$

[*Benares*, 1935]

11. If $w_1, w_2, w_3$ be the apparent weights of a given body in fluids whose sp. gravities are $s_1$, $s_2$ and $s_3$ respectively, then show that

$$w_1 (s_2 - s_3) + w_2 (s_3 - s_1) + w_3 (s_1 - s_2) = 0.$$ [*Agra*, 1933]

12. A body immersed in a liquid is balanced by a weight $P$ to which it is attached by a thread passing over a fixed pulley, and when half immersed, is balanced in the same manner by a weight $2P$. Prove that the densities of the body and liquid are as 3 : 2.

13. A solid whose sp. gr. is 1·85 is weighed in a mixture of alcohol of sp. gr. 0·82 and water. It weighs 28·8 grammes in vacuum and 14·1 grammes in the mixture; find the proportion of alcohol present.

14. A body of sp. gr. $\sigma$ when weighed against a weight of sp. gr. $\varrho$ in water—the whole balance being immersed—appears to have a weight $W$. Show that its true weight is

$$\frac{\sigma}{\sigma - 1} \cdot \frac{\varrho - 1}{\varrho} \cdot W.$$

**5·6. Bodies floating under constraint.** A body wholly or partialy immersed in a liquid may not be floating freely; it may be doing so under some constraint. For example, it may have a fixed point about which it may be free to turn, or it may have a supporting string attached to some point of it. Again if the body is immersed in a liquid whose sp. gr. is greater than that of the body, it will require some constraining force to prevent it from ascending. In such cases, over and above the weight of the body and the force of buoyancy, there is also some external force brought into play, and for equilibrium of the body, we have to take into consideration all these three types of forces.

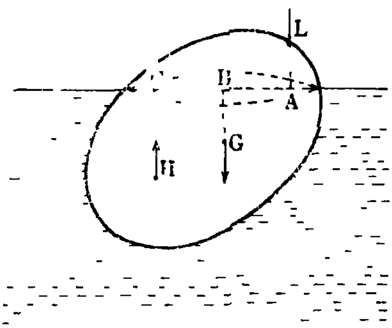
**5·61. A body immersed in a liquid whose sp. gr. is greater than that of the body.** If a body be wholly immersed in a liquid whose sp. gr. is greater than that of the body, then the force of buoyancy will be greater than the weight of the body. The body will therefore ascend unless some downward force is applied to prevent it from doing so.

If $T$ denotes the downward tensional force, $W$ the weight of the body and $W'$ the weight of the liquid displaced, then, applying the method of 5·4, we shall get the following relation :—

$$T = W' - W.$$

**5·62. The Balloon.** The foregoing principle may be illustrated by the case of a balloon. A balloon generally consists of a large nearly spherical envelope, made of silk or some other strong and light material. It is filled with a gas of less density than air, like coal gas or hydrogen. A light car for accommodating one or two persons may also be attached to the balloon. The size of the balloon is so adjusted that the weight of the air displaced is greater than the combined weight of the balloon and the car holding the aeronauts. The balloon thus ascends, and will continue ascending until the weight of the air displaced becomes equal to the weight of the balloon and the car.

**5·63. A body floating with a string attached to a point of it.** Suppose a body is floating, partly immersed in a liquid and supported by a string attached to the point $L$ of the body.




Let $T$ be the upward tension of the string and $W$ the weight of the body acting downwards through the centre of gravity $G$. Let $V$ be the volume of the liquid displaced and $w$ the weight of unit volume of the liquid, so that the force of buoyancy is $Vw$ acting upwards through the centre of buoyancy $H$.

The vertical forces acting on the body are only these three, $T$, $W$, $Vw$; and for equilibrium they must be in the same vertical plane. Suppose the verticals through $L$, $G$ and $H$ meet the horizontal line $ABC$ in the surface in the points $A$, $B$ and $C$ respectively.

*The conditions of equilibrium are, therefore, the following* :—

$$T + Vw = W, \quad \ldots \ldots \quad (1)$$

$$Vw \times AC = W \times AB. \quad \ldots \ldots \quad (2)$$

**5·64. Body turning about a fixed point.** If a floating body be free to turn about a fixed point, then there will be a force of reaction at the fixed point. The other two forces, the weight of the body and the force of buoyancy, being vertical, the reaction also will be vertical, and the conditions of equilibrium will be similar to those obtained in the preceding article

The reaction at the fixed point will be obviously equal to the difference of the weight of the body and the force of buoyancy, and will act upwards.

**5·7. Illustrative Examples.** (i) *A balloon of volume $V$, contains a gas whose density is to that of the air at the earth's surface as 1 : 15. If the envelope of the balloon be of weight $w$ but of negligible volume, find the acceleration with which it will begin to ascend.*

If $\sigma$ be the density of the air at the earth's surface, that of the gas is $\frac{1}{15}\sigma$.

The total mass of the balloon

$$=\tfrac{1}{15}\sigma V+w/g$$

If $f$ be the required acceleration, we have

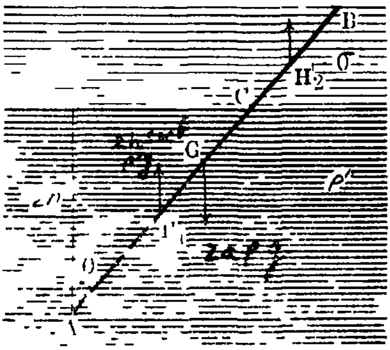
$$(\tfrac{1}{15}\sigma V+w/g)f=\text{the upward force of buoyancy}$$
$$-\text{wt. of the balloon}$$
$$=V\sigma g-(\tfrac{1}{15}\sigma Vg+w)=\tfrac{14}{15}\sigma Vg-w,$$

or $$f=\frac{14V\sigma g-15w}{V\sigma g+15w}g.$$

(ii) *A uniform rod of length $2a$ and density $\rho$, is moveable in a vertical plane about one end which is fixed in a liquid of density $\rho'$ at a depth $2b$ below the surface. A liquid of smaller density $\sigma$ is added on the top of the first liquid; if in the oblique position of equilibrium the rod is just covered by the liquid, prove that its inclination to the vertical is*

$$\cos^{-1}\left[\frac{b}{a}\left(\frac{\rho'-\sigma}{\rho-\sigma}\right)^{1/2}\right].$$ [*Nagpur*, 1942]

Let $2a$ be the length of the rod $AB$, the end $A$ being hinged at a depth $2b$ below the surface of the liquid of density $\varrho'$. Let $G$, $H_1$, $H_2$ be the middle points of $AB$, $AC$, $CB$ respectively, so that $G$ is the C. G. of the rod and $H_1$ and $H_2$ the centres of buoyancy due to the liquids of densities $\varrho'$ and $\sigma$ respectively. Let $\theta$ be the inclination of the rod to the vertical.




The lengths $AC$, $CB$ of the rod in the lower and upper liquids are respectively $2b \sec\theta$ and $(2a - 2b\sec\theta)$.

Apart from the reaction at the hinge, there are three vertical forces acting on the rod. They are :

(1) The weight of the rod $2a\varrho g$ acting downwards at $G$.

(2) The force of buoyancy $2b \sec\theta\, \varrho' g$ due to the lower liquid acting upwards through $H_1$.

(3) The force of buoyancy $(2a - 2b\sec\theta)\sigma g$ due to the upper liquid acting upwards through $H_2$.

Now taking moments of the forces about the point $A$, we get

$$2a\varrho g \times a = 2b\sec\theta\varrho' g \times b\sec\theta + (2a - 2b\sec\theta)\sigma g \times (a + b\sec\theta),$$

or $$a^2\varrho = b^2\sec^2\theta\varrho' + (a^2 - b^2\sec^2\theta)\sigma,$$

or $$b^2\sec^2\theta(\varrho' - \sigma) = a^2(\varrho - \sigma),$$

or $$\cos\theta = \frac{b}{a}\cdot\sqrt{\frac{\varrho' - \sigma}{\varrho - \sigma}},$$

or $$\theta = \cos^{-1}\left\{\frac{b}{a}\left(\frac{\varrho' - \sigma}{\varrho - \sigma}\right)^{1/2}\right\}.$$

## Examples XII

1. Two cubic feet of cork of sp. gr. 0·24 is kept below water by a rope fastened to the bottom. Prove that the tension of the rope is 95 pounds.

2. The mass of a balloon and its car is 3000 lbs., the mass of air displaced is 3400 lbs.; with what acceleration does the balloon rise?

3. The mass of a balloon and the gas it contains is 3500 lbs. If the balloon displace 48000 cu. ft. of air and the mass of a cu. ft. of air be 1·25 ozs., find the acceleration with which the balloon commences to ascend.

4. A uniform rod capable of turning about one end, which is out of the water, rests inclined to the vertical with one-third of its length in some water. Prove that its sp. gr. is $\frac{5}{9}$. [*Calcutta*, 1938]

5. A uniform rod rests in a position inclined to the vertical, with half its length immersed in water, and can turn about a point in it at a distance equal to one-sixth of the length of the rod from the extremity below the water. Prove that the sp. gr. of the rod is 0·125.

6. A thin rod, of sp. gr. $\frac{3}{4}$ and of length 4 ft. floats partly immersed in water, being supported by a string fastened to one end of the rod, how much of the rod is immersed? If the upper end of the rod is 1 ft above the surface of the water, at what angle is the rod inclined to the horizontal? [*Allahabad*, 1922]

7. A uniform rod of length $2a$, can turn freely about one end which is fixed at a height $h(<2a)$ above the surface of a liquid, if the densities of the rod and liquid be $\rho$ and $\sigma$, show that the rod can rest either in a vertical position or inclined at an angle $\theta$ to the vertical such that

$$\cos\theta = \frac{h}{2a}\sqrt{\frac{\sigma}{\sigma-\rho}}.$$

[*Lucknow*, 1932, 1937; *Agra* 1931, 1928, 1943; *Allahabad*, 1935]

8. A solid hemisphere floats in a liquid, completely immersed with a point of the rim joined to a fixed point by means of a string. Find the inclination of the base to the vertical and the tension of the string, $\rho$, $\sigma$ being the densities of the solid and the liquid.

9. A semicircular cylinder floats in water with its axis fixed in the surface of water. If this cylinder be moveable about the fixed axis and if its density be half of that of water, show that it will be in equilibrium in any position. [*Agra*, 1939]

10. An equilateral triangular lamina suspended freely from $A$, rests with the side $AB$ vertical and the side $AC$ bisected by the surface of a heavy fluid; prove that the density of the lamina is to that of the fluid as 15 : 16.

11. A triangular lamina $ABC$, of which the sides $AB$, $AC$ are equal, floats in water with $BC$ vertical, and three quarters of its length immersed, being kept in equilibrium in this position by means of a

string fastened to $A$ and the bottom of the vessel. Find the sp. gr. of the lamina, and show that the tension of the string is $\frac{1}{27}$th of the weight of the lamina.

## Examples XIII

1. A thin uniform rod of weight $W$ has a particle of weight $w$ attached to one end. It is floating, in an inclined position in water, with this end immersed. Prove that the length of the rod above water is $w/(w+W)$ times its whole length and that the sp. gr of the rod is

$$W^2/(w+W)^2.$$

2. A cone of given weight and volume, floats in a given fluid with its vertex downwards, show that the surface of the cone in contact with the fluid is least, when the vertical angle of the cone is $2\tan^{-1}(1/\sqrt{2})$

3. A solid hemisphere of radius $a$ and weight $W$ is floating in liquid, and at a point on the base at a distance $c$ from the centre rests a weight $w$, show that the tangent of the inclination of the axis of the hemisphere to the vertical for the corresponding position of equilibrium, assuming the base of the hemisphere to be entirely out of the fluid, is

$$\frac{8}{3}\frac{c}{a}\cdot\frac{w}{W}.$$

4. A thin hollow cone, with a base, floats completely immersed in water wherever it is placed; show that the vertical angle is

$2\sin^{-1}\frac{1}{3}$. [*Agra*, 1928; *Allahabad*, 1937]

5. A hollow closed cone of semi-vertical angle $\sin^{-1}\frac{1}{3}$ of metal whose specific weight is $w'$ is made of such uniform thickness that it will float in all positions wholly submerged in liquid of specific weight $w$. Show that the thickness must be

$$\tfrac{1}{4}h\{1-(1-w/w')^{1/3}\},$$

where $h$ is the external height of the cone. [*Nagpur*, 1943]

6. A rod floats upright partially immersed in a homogeneous liquid. Prove that a small increase of atmospheric density will produce a small rise of the rod proportional to the square of the length of the unimmersed portion. [*Allahabad*, 1943]

7. A rod of density $\rho$ and length $a$ is freely moveable about one end fixed at a depth $c$ below the surface of a liquid of density $\sigma$; show that the rod may rest in a position inclined to the vertical provided that

$$\sigma/\rho > 1 < a^2/c^2.$$

8. A triangular lamina $ABC$ of sp. gr. $s$ floats with its plane vertical in water, $A$ being outside the liquid and $BC$ not horizontal. The angle $A$ is a right angle and $AB = AC$. If $\theta$ be the inclination of $AB$ to the vertical, prove that

$$\sin 2\theta = (2 - 2s)/(2s - 1), \text{ given } s > \tfrac{3}{4}. \quad [\textit{Nagpur}, 1935]$$

9. A uniform isosceles triangular lamina of sp. gr. $\sigma$ floats in water with its plane vertical, its vertical angle ($= 2\alpha$) immersed and the base wholly above water. Prove that in the position of equilibrium in which the base is not horizontal, the sum of the lengths of the immersed portions of the two sides is $2a \cos^2\alpha$ where $a$ is the length of one of the equal sides; and that $\sigma$ is less than $\cos^4\alpha$ as well as $\cos 2\alpha$.

[Hint :—If $AD$, $AE$ be the lengths immersed, for the second part use the condition that $(AD + AE) > 2\sqrt{AD.AE}$, and to get the other inequality take the extreme case when the base $BC$ is just out of the fluid.]

10. A cylindrical piece ($B$) of material of sp. gr. $\varrho(>1)$, height $H$, and cross-sectional area $\beta$ rests with its axis vertical at the bottom of a cylindrical vessel ($A$) of cross-sectional area $\alpha$. A liquid of sp. gr. $\sigma$ ($>\varrho$) is poured into $A$ to a height $h$ such that $B$ does not rise. If now water be poured into $A$ until $B$ is completely immersed, show that $B$ will have risen a height $x$, provided $x$ is positive, where $x$ is

$$\left(1 - \frac{\beta}{\alpha}\right)\left(h - \frac{\varrho - 1}{\sigma - 1} \cdot H\right). \qquad [\textit{Allahabad}, 1926]$$

11. A cylindrical vessel $A$, the area of whose cross-section is $\alpha$ sq. cm., is placed with its base on a horizontal table. An iron cylinder ($B$) whose height is $H$ cm. and sp. gr. 7·5, and the area of whose cross-section is $\beta$ sq. cm., rests with its axis vertical on the bottom of $A$. Mercury (sp. gr. 13·5) is now poured into $A$ to a depth $h$ cm. Show that $B$ will not rise so long as $5H > 9h$. Water is now poured into $A$ until $B$ is immersed. Prove that $B$ will have risen a height

$$(1 - \beta/\alpha)(h - 13H/25) \text{ cm.},$$

provided this expression is positive. [*M.T.*]

12. A bucket half-full of water is suspended by a string which passes over a pulley small enough to let the other end fall into the bucket. To this end is tied a ball whose sp. gr. $\sigma$ is greater than 2. Show that, if the ball does not touch the bottom of the bucket and if no water overflows, equilibrium is possible if the weight of the ball lie between $W$ and $\sigma W/(\sigma - 2)$, where $W$ is the weight of the bucket and water. [*Benares*, 1938]

[Suppose $V$ is the volume of sphere, $\gamma V$ the volume immersed and $w$ the weight of a unit volume of water. If $T$ be the tension of the string, $T = V\sigma w - \gamma V w$, where $(0 \leqslant \gamma \leqslant 1)$.

Also $T = W + \gamma V w$.

$$\therefore \quad W = V\sigma w - 2\gamma V w$$

$$= V\sigma w(1 - 2\gamma/\sigma).$$

$$\therefore \quad V\sigma\omega = \sigma W(\sigma - 2\gamma).$$

The max. and min. values are given by puttig $\gamma = 0$ and $\gamma = 1$. Hence the weight lies between $W$ and $\sigma W/(\sigma - 2)$.]

13. A cylindrical bucket with water in it balances a mass $M$ over a pulley. A piece of cork, of mass $m$ and sp. gr. $\sigma$, is then tied to the bottom of the bucket so as to be totally immersed. Prove that the tension of the string will be

$$\frac{2Mmg}{2M + m}\left(\frac{1}{\sigma} - 1\right), \qquad [M. T.; U. P. C. S., 1939]$$

and that the pressure on the curved surface of the bucket will be greater or less than before according as the volume of the cork has to the volume of the water a ratio greater or less than

$$\left(\sqrt{1 + \frac{m}{2M}} - 1\right) : 1 .$$

[The mass on one side of the pulley is $M$ and on the other $M + m$, hence the latter will descend with a constant acceleration $f$ equal to

$$g\frac{m}{m + 2M}.$$

Considering the equilibrium of the cork, forces acting upon it are

(1) its weight $mg$, downwards,

(2) the tension $T$ of the string, downwards,

(3) the upward thrust.

$$\therefore \quad mf = mg + T - \text{the upward thrust.} \qquad \ldots \quad (1)$$

Now the mass of the cork is $m$. Hence the mass of the water displaced is $m/\sigma$. When the mass is falling with an acceleration $f$, the effective weight (as would be shown, for example, by a spring balance) will be $(m/\sigma)(g - f)$. Hence the upward thrust on the cork

$$= (m/\sigma)(g - f).$$

Substituting in (1), we get

$$m\left(1 - \frac{1}{\sigma}\right)f = mg\left(1 - \frac{1}{\sigma}\right) + T,$$

or
$$T = \left(1 - \frac{1}{\sigma}\right)m(f - g)$$
$$= \left(\frac{1}{\sigma} - 1\right)\frac{2Mmg}{2M + m}.$$

For the second part, if $V$ and $v$ be the volumes of the water and cork respectively and $h$ and $H$ the depths of water before and after the cork has been introduced, then
$$h : H = V : (v + V).$$

The pressure originally on the curved surface was $m_1 g . 2\pi ah . \frac{1}{2}h$, where $m_1$ is the mass of unit volume of water.

In the second case $h$ has to be replaced by $H$, and $g$ has to be replaced by $g - f$, for now the effective weight (which produces the pressure) per unit volume of water is $m_1 (g - f)$. Hence the new pressure is
$$m_1 (g - f) . 2\pi aH . \tfrac{1}{2}H, \text{ i.e., } m_1 g \left(\frac{2M}{m + 2M}\right) . 2\pi aH . \tfrac{1}{2}H.$$

Hence the new pressure $\gtrless$ the old pressure according as
$$\frac{2MH^2}{2M + m} \gtrless h^2,$$
or
$$\frac{H}{h} \gtrless \sqrt{1 + \frac{m}{2M}},$$
or
$$\frac{v + V}{V} \gtrless \sqrt{1 + \frac{m}{2M}},$$
or
$$\frac{v}{V} \gtrless \sqrt{\left(1 + \frac{m}{2M}\right)} - 1.]$$

14. Two buckets containing water, the mass of each bucket with the contained water being $M$, balance each other over a smooth pulley. Two pieces of wood of masses $m$, $m'$ and specific gravities $\sigma$, $\sigma'$ are then tied to the bottoms of the buckets so as to be wholly immersed; prove that the tension of the string attached to the mass $m$ is
$$\frac{2m(M + m')g}{2M + m + m'}\left(\frac{1}{\sigma} - 1\right).$$

15. If a body floats in liquid with volumes $v_1$, $v_2$ and $v_3$ above the surface when the barometric heights are $h_1$, $h_2$, and $h_3$, prove that
$$h_1 v_1(v_2 - v_3) + h_2 v_2(v_3 - v_1) + h_3 v_3(v_1 - v_2) = 0. \quad [\textit{Calcutta}, 1914]$$

16. A body floating in water has volumes $v_1, v_2, v_3$ above the surface when the densities of the surrounding air are respectively $\rho_1, \rho_2, \rho_3$. Prove that

$$\frac{\rho_2 - \rho_3}{v_1} + \frac{\rho_3 - \rho_1}{v_2} + \frac{\rho_1 - \rho_2}{v_3} = 0. \qquad [\textit{Agra}, 1937]$$

17. Prove that, if volumes $v$ and $V$ of two different substances balance in vacuum, and volumes $v'$, $V'$ balance when weighed in liquid, the densities of the substances and the liquid are as

$$\frac{v' - V'}{v} : \frac{v' - V'}{V} : \left(\frac{v'}{v} - \frac{V'}{V}\right).$$

18. A substance whose density is $\rho$, is weighed by means of weights, the density of which is $\rho'$; if $\sigma$ be the density of the air, find what is the true weight corresponding to any apparent weight. If the density of the air increases from $\sigma$ to $\sigma'$, prove that the apparent weight of the body is less than its former weight by a fraction

$$\frac{(\rho' - \rho)(\sigma' - \sigma)}{(\rho - \sigma)(\rho' - \sigma')}$$

of the latter, $\rho'$ being greater than $\rho$. [*Allahabad*, 1932]

19. A rectangle moveable about an angular point floats with half its area immersed in a liquid. If the angular point lie outside the liquid, and if the rectangle float with its sides equally inclined to the vertical, show that the ratio of the density of the rectangle to that of the liquid is $3b + a : 4b$, where $a$, $b$ are the sides of the rectangle.

20. A rectangle, moveable about an angular point which is fixed below the surface of a liquid, floats with its sides equally inclined to the vertical and with half its area immersed in the liquid. If the lengths of the sides be $a$ and $b$ and one of the sides of length $b$ be entirely immersed in the liquid, show that the ratio of the density of the body to that of the liquid is $a - b : 4a$.

21. A uniform log of square section floats in water with one angle below the surface; prove that there are two unsymmetric positions of equilibrium, if the sp. gr. of the log be less than $9/32$.

CHAPTER VI

# GASES

**6·1. Some general properties.** It has been pointed out in Chap. I that the chief characteristic difference between a liquid and a gas is that a liquid is practically incompressible whereas a gas is highly compressible and elastic. Gases are also marked for their low density and great expansion when heated. Air is the most familiar example of a gas just as water is that of a liquid.

There are certain properties of gases with which every student must be familiar at this stage, and we need not describe here the experiments by means of which those properties were first discovered or may be demonstrated in the laboratory. Every one knows now that air has weight, but curiously enough, this important property of the heaviness of air remained unknown till comparatively modern times. It is said that Aristotle (B. C. 384-322) suspecting the truth of it, wanted to verify experimentally if air has any weight. He weighed a sheep skin first empty and then inflated with air, and finding that the weight was the same in both cases, he concluded that the air was a weightless fluid! For many centuries the conclusion of Aristotle's experiment was regarded as decisive and it was not till the time of Galileo (1564-1642) that air was generally accepted to be a heavy fluid. In fact it was Otto von Guericke of Magdeburg (1602-1686), the inventor of air-pumps, who for the first time conclusively demonstrated that *air has weight*, by performing his well-known experiment of weighing a flask first full of air and then exhausted of air.

Again, gases exert pressure as do liquids, and the pressure of a gas is measured in the same way as that of a liquid. In the case of a liquid, however, the pressure is entirely due to its weight and the external pressure, if any, while the pressure of a gas, though affected by the action of gravity, depends chiefly upon its volume and temperature.

It is also known experimentally that the density of a given quantity of gas does not remain invariable; in fact it changes with the change of its temperature and pressure. In order to compare the densities and specific gravities of gases it is, therefore, necessary that they should be reduced to the same temperature and pressure. For this purpose the

standard temperature is usually taken to be 0° C. and the standard pressure to be the pressure due to a column of 76 cms. of mercury.

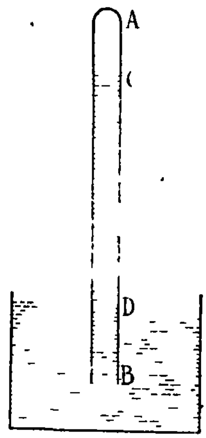
**6·11. Specific gravities of some gases.** At temperature 0°C. and pressure 76 cms. of mercury, the following are the specific gravities of some of the common gases, water being the standard substance :—

| | |
|---|---|
| Air .. .. .. .. | 0·001293, |
| Oxygen .. .. .. .. | 0·001430, |
| Hydrogen ... .. .. .. | 0·000089, |
| Nitrogen .. .. .. .. | 0·001256, |
| Carbon dioxide .. .. .. | 0·001977. |

It appears from above that Hydrogen is much lighter than air, it is about one-fourteenth as heavy as air. In fact hydrogen is the lightest gas known and hence for measuring sp. gr. of gases, it is found more convenient to take hydrogen as the standard substance than water.

**6·12. Pressure of the Atmosphere. Torricelli's Experiment.** In 1643, Evangelista Torricelli (1608-1647), a pupil of Galileo, performed a simple but important experiment to determine the measure of the atmospheric pressure.

A uniform straight glass tube $AB$ about 34 inches in length, closed at one end and open at the other, is filled with mercury. Then closing the open end with the thumb, the tube is inverted and placed in a trough of mercury with the temporarily closed end $B$ immersed in it. On removing the thumb, it is found that mercury descends through a certain length leaving a vacuum $AC$ at the top of the tube with its upper surface $C$ at a height of about 29 or 30 inches from the level $D$ which is a point inside the tube



in the horizontal plane of the mercury surface in the trough.

Now, by **2·2**, the pressure inside the tube at level $D$ is the same as the pressure outside the tube at the surface of mercury. The pressure inside the tube at $D$ is due to a column $CD$ of mercury, i.e., $g\sigma.CD$, where $\sigma$ is the density of mercury. The pressure outside the tube at the surface of mercury is the pressure due to the atmosphere. Hence the atmospheric pressure is $g\sigma.CD$.

*The space above the surface of mercury in the tube is called the* **Torricellian vacuum.**

Strictly speaking there is some mercury vapour in the space $AC$; but the pressure produced by it being negligibly small, the pressure of the atmosphere may be taken for all practical purposes to be $g\sigma.CD$.

If Torricelli's experiment be performed, it will be noticed that the height $CD$ of mercury does not remain constant; it is continually changing. It follows, therefore, that the pressure of the atmosphere is not constant.

**6·13. Height of the homogeneous atmosphere.** The atmosphere is not homogeneous; its density gradually decreases as one proceeds up from the surface of the earth But supposing that the atmosphere is replaced by a column of homogeneous air of the same density as at the surface of the earth and of height such that the pressure produced by this column is the same as the actual atmospheric pressure, then this height of the column of air is called the *height of the homogeneous atmosphere.*

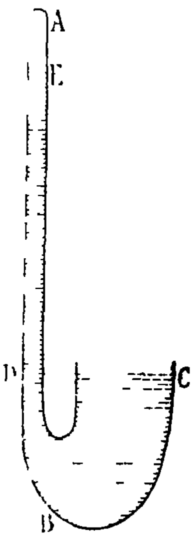
The height of the homogeneous atmosphere comes out to be about 5 miles.

**6·2. The Barometer.** The barometer is an instrument which is used for measuring the atmospheric

pressure. The glass tube apparatus used in Torricelli's experiment constitutes the simplest type of a barometer. Any liquid instead of mercury may be used. But mercury, on account of its high specific gravity which is nearly 13·6, is the best liquid for this purpose, otherwise the tube will have to be inconveniently long. For instance, if water be used in place of mercury, the tube must be more than 33 feet long.

**6·21. Siphon Barometer.** A form of barometer very commonly in use is known as the *siphon barometer.*

It consists of a bent tube $ABC$, closed at $A$ and open at the end $C$. The height of the long part $AB$ is usually about 32 or 33 inches and the cross-section of this part is much smaller than that of $BC$. Mercury having been put into the tube in sufficient quantity to fill the longer limb, the tube is placed in a vertical position when the mercury assumes a position of equilibrium as shown in the figure, the space $AE$ above the mercury being vacuum.




If the horizontal plane of the mercury surface in $BC$ intersects $AB$ in $D$, then as explained in **6·12**, the pressure of the atmosphere is $g\sigma.ED$, where $\sigma$ is the density of mercury.

The height of the barometric column $ED$ generally varies between $29\frac{1}{2}$ and 30 inches of mercury or about 76 centimetres. The corresponding atmospheric pressure is about 14·7 lbs. per square inch.

**6·22. Graduation of the Barometer.** In taking the readings of the siphon barometer (figure of **6·21**)

we must keep in view the fact that if the mercury level in $BA$ rises, its level in $BC$ must fall and the height of the barometric column is the difference of these two levels.

Suppose $k$ and $K$ are the sectional areas of $BA$ and $BC$ respectively, and suppose the mercury level in $BA$ rises (falls) by $x$; then the fall (rise) of the level in $BC$ will be by $\frac{k}{K}x$.

Thus an apparent variation of $x$ in the barometric column would correspond to a real variation of

$$x + \frac{k}{K}x, \quad \text{i.e., of } \frac{k+K}{K}x.$$

Hence an apparent variation of $\frac{K}{k+K}x$ would correspond to a real variation of $x$.

For this reason, in the graduation of a barometer the distances between successive markings in the long tube are generally kept shorter than they are marked in the ratio $K : k + K$

Thus, in particular, if $k$ and $K$ be $1/10$ and $1$ sq. inch respectively, then the distance given as one inch by the graduations on the long tube, must really be kept only $10/11$ inch long.

**6·23. Corrections to the Barometer reading.** In order to ensure a good degree of accuracy in the barometric readings, it is necessary to apply several types of corrections to the observations taken. For a detailed description of these corrections reference may be made to any standard book on Physics. In this book, we contend ourselves by simply indicating the principal types of corrections which are usually applied.

(i) *Correction for capacity.*

If the barometer be not graduated as indicated in **6·22,** but the graduations be marked at their actual distances, then the reading of the apparent variation must be multiplied by $(k + K)/K$ to give the true variation.

(ii) *Correction for temperature.*

Since the mercury and the measuring rod both expand with heat, account must be taken of the temperature at which observations are made.

(iii) *Correction for capillarity.*

When the mercury is contained in a narrow tube, the capillary action has a depressing effect upon the mercury column, so that the top of the mercury in the tube is not flat but is convex.

(iv) *Correction for vapour pressure.*

In the Torricellian vacuum there is some mercury vapour the pressure of which depresses the column, but its effect is extremely small.

(v) *Correction for unequal value of gravity.*

The value of $g$ is not the same everywhere; it depends on the latitude and on the elevation of the place of observation. Hence, when barometric observations extending over a wide area are compared, a correction for unequal value of $g$ is required to be applied.

**6·3. Relation between the Pressure and Volume of a gas. Boyle's Law.** The important *experimental* law, given below expresses the relation between the pressure and volume of a gas, when the temperature is constant. This law, known as *Boyle's law* in England, was enunciated by Robert Boyle (1626-1691) in 1662 and can be verified experimentally. On the continent it is generally attributed to Marriotte and is called *Marriotte's law.*

## Boyle's Law

**If the temperature of a given mass of gas remains constant, the pressure varies inversely as its volume; i.e.,**

$$pv = \text{const.},$$

**where p denotes the pressure of the gas and v its volume.**

**6·31. Variations from Boyle's Law.** By experiments more carefully performed it has been observed that Boyle's law is not perfectly true. For gases like air, oxygen, hydrogen, and nitrogen, which are very hard to liquefy, the law is very nearly true for a

considerable range of pressures and temperatures. But for easily liquefiable gases, such as carbon dioxide and ammonia, the volume is found to decrease more rapidly than the pressure increases and the deviation from the Boyle's law is quite appreciable.

**6·32. Perfect Gas.** *A gas which absolutely obeys Boyle's law is called a perfect gas.*

We have just seen that there are no perfect gases; air, oxygen, hydrogen and nitrogen may be regarded as very approximately perfect gases. In fact a *perfect gas* is only an ideal substance for which Boyle's law is assumed to be true always.

In this book it is assumed, unless otherwise stated, that every gas under discussion is a perfect gas.

**6·33. Relation between Pressure and Density.** Let $p$, $v$, and $\rho$ denote respectively the pressure, volume, and density of a given mass of gas. When the pressure is changed from $p$ to $p'$, suppose $v$ and $\rho$ become $v'$ and $\rho'$, the temperature remaining unaltered.

Now, by Boyle's law, we have

$$pv = p'v'. \qquad \ldots \quad (1)$$

Again, since the mass of the gas remains unaltered, we get

$$\rho v = \rho' v'. \qquad \ldots \quad (2)$$

From (1) and (2), we get

$$\mathbf{p/\rho = p'/\rho' = const.}, \qquad \ldots \quad (3)$$

or

$$\mathbf{p = k\rho}, \qquad \ldots \quad (4)$$

where $K$ is a constant.

Hence, *if the temperature remains unaltered, the pressure of a gas is proportional to its density.*

**6·4. Effect on Volume of a change of Temperature. Charles' Law.** The volume of a given mass of gas changes considerably with change of temperature. The

relation connecting the rise of temperature and the increase of volume is expressed by an *experimental* law, known as *Charles' law*, which is said to have been first obtained by J. A. C. Charles in 1787. The discovery of this law is also attributed to Dalton and Gay-Lussac who published it, independently of each other, in 1801 and 1802 respectively.

### Charles' Law

**If the pressure of a given mass of gas remains constant, the volume of the gas increases by a definite fraction $\alpha$ of its volume at 0°C. for every degree centigrade of rise in the temperature.**

Thus, if $V_0$ be the volume of the gas at 0°C. and $V$ its volume at $t$°C., then

$$V - V_0 = V_0.\alpha.t,$$

or

$$\mathbf{V = V_0(1 + \alpha t).} \quad \ldots \quad (1)$$

For air and most of the gases, the value of $\alpha$, which is called the *coefficient of expansion*, is 0·003665 or 1/273 approximately.

Like Boyle's law, this law also can be verified by experiments for the descriptions of which reference may be made to books on Physics.

If $\varrho_0$ and $\varrho$ be the densities of the gas at temperatures 0°C. and $t$°C, respectively, then, since

$$\text{mass of the gas} = \varrho_0 V_0 = \varrho V,$$

we have

$$\varrho_0/\varrho = V/V_0 = 1 + \alpha t \text{ from (1)},$$

or

$$\varrho_0 = \varrho\,(1 + \alpha t). \quad \ldots \quad (2)$$

**6·41. Relation between Pressure, Density and Temperature.** Suppose the pressure and density of a given mass of gas at 0°C. temperature are $p$ and $\varrho_0$ respectively. Then from (4) of **6·33**, we have

$$p = K\varrho_0, \quad \ldots \quad (1)$$

where $K$ is a constant depending upon the nature of the gas.

Now if keeping the pressure $p$ unaltered, the temperature of the gas be raised to $t$°C., so that the density changes from $\rho_0$ to $\rho$, then from (2) of 6·4, we get

$$\rho_0 = \rho(1 + \alpha t).$$

Putting this value of $\rho_0$ in (1), we have finally

$$\mathbf{p = K\rho(1 + \alpha t).} \quad . \quad . \quad . \quad (2)$$

**6·42. Absolute Temperature.** Suppose a gas without liquefying can be continually cooled down till its temperature is far below 0°C. and that it continues to obey Charles' law, then it is clear from (2) of the preceding article that its pressure will vanish, without any change in its volume, at a temperature $t$ such that

$$1 + \alpha t = 0,$$

or

$$t = -1/\alpha = -273°\text{C}.$$

This temperature, $-273$°C., is called the **absolute zero,** and the temperatures measured from this zero point are called **absolute temperatures.**

The absolute temperature is generally denoted by $T$, so that corresponding to $t$°C., the absolute temperature

$$T = 1/\alpha + t = 273 + t.$$

**6·43. Relation between Pressure, Volume and Absolute Temperature.** From (2) of 6·41, we get

$$p = K\rho(1 + \alpha t) = K\rho\alpha(1/\alpha + t) = K\rho\alpha T,$$

or

$$p/T = K\rho\alpha.$$

If $V$ be the volume of a given mass of the gas, then

$$pV/T = K\rho\alpha V = K\alpha(V.\rho) = K\alpha \times \text{mass of the gas},$$

or

$$pV/T = \text{const.} \quad . \quad . \quad . \quad (1)$$

Hence, *if the pressure, volume and absolute temperature of a given mass of gas change from p, V, T to p′, V′, T′ respectively, then*

$$\mathbf{pV/T = p'V'/T'.} \quad . \quad . \quad . \quad (2)$$

**6·5. Mixture of Gases.** The following property of gases is deduced from experimental observations :—

*If two gases contained in two closed vessels of different volumes, be at the same temperature and pressure, and if a communication is opened between the two vessels, the gases form a mixture which has the same pressure and temperature as the constituent gases, provided no chemical action takes place between them.*

The above experimental fact enables us to establish the following proposition:—

**If the pressures of two gases at the same temperature and each of volume v, be $p_1$ and $p_2$, then the pressure of the mixture of the two gases, when the combined volume is v, is $p_1 + p_2$, the temperature remaining unaltered.**

To prove this, suppose the pressure of the second gas is changed from $p_2$ to $p_1$ before mixing, so that its volume alters from $v$ to $v'$. Then, by Boyle's law,

$$p_1 v' = p_2 v. \quad \quad (1)$$

When the gases are mixed, the volume of the mixture is $v + v'$ and by the experimental law stated above, its pressure is $p_1$. Now let the volume of the mixture be changed to $v$ and suppose the pressure then becomes $P$.

Applying Boyle's law, we get

$$Pv = p_1(v + v') = p_1 v + p_2 v \text{ from (1)},$$

or

$$\mathbf{P = p_1 + p_2}. \quad \quad (2)$$

Clearly a similar result will hold true for a mixture of any number of gases. The result is known as *Dalton's Law* for the pressure of a mixture of gases.

**6·51. Pressure of a mixture of gases of different volumes.** *If two gases of volume $v_1$ and $v_2$ and pressures $p_1$ and $p_2$ respectively be mixed together to form a mixture of*

*volume $V$, to find the pressure of the mixture, assuming the temperature to remain throughout the same.*

Let us suppose that before the gases are mixed, the volume of each is changed to $V$; then, by Boyle's law, their pressures will be respectively

$$\frac{p_1v_1}{V} \text{ and } \frac{p_2v_2}{V}.$$

Now, if the gases be mixed so that the volume of the mixture is $V$, then by the proposition of **6·5**, the pressure $P$ of the mixture is given by

$$P = \frac{p_1v_1}{V} + \frac{p_2v_2}{V},$$

or
$$\mathbf{P} = \frac{\mathbf{p_1v_1 + p_2v_2}}{\mathbf{V}}. \quad . \quad . \quad (1)$$

The above result can easily be extended to a mixture of more than two gases.

**6·6. Illustrative Examples.** (i) *Calculate the height of the homogeneous atmosphere at zero centigrade, when the height of the mercurial barometer is* 76 *cms. and the sp. gr. of mercury and air are* 13·6 *and* 0·001293 *respectively.*

If the height of the homogeneous atmosphere be $h$ cm., the pressure on one sq. cm.

$$= h \times 0·001293 \text{ grammes.}$$
$$= 76 \times 13·6 \text{ grammes.}$$

$$\therefore \qquad h = \frac{76 \times 13·6}{0·001293} \text{ cm.}$$
$$= \frac{10276000}{1293} \text{ meters}$$
$$= 7947·33 \ldots \text{ meters.}$$

(ii) *A bubble of gas* 100 *c. mm. in volume is formed at a depth of* 100 *meters in water; find its volume when it reaches the surface, the height of the barometer being* 76 *cm. and the density of mercury* 13 6.

The height of the *water* barometer

$$= 76 \times 13·6 \text{ cm.} = 10·34 \text{ meters approx.}$$

The pressure at a depth of 100 meters below the surface of water is due to a column of water $= (10 \cdot 34 + 100)$ meters.

If $V$ c. mm. be the volume of the bubble at the surface of water, we get by applying Boyle's law,

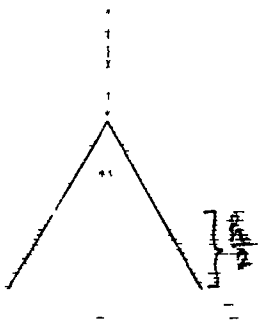
$$V \times 10 \cdot 34 = 100 \times 110 \cdot 34,$$

or
$$V = \frac{100 \times 110 \cdot 34}{10 \cdot 34} \text{ c. mm.}$$
$$= 1067 \text{ c. mm. approx.}$$

(iii) *A hollow cone is immersed mouth downwards in water the surface of which is exposed to the atmospheric pressure. Show how far it may be depressed so that the water within the cone may rise half-way up it.*

[*Calcutta*, 1916]

Let $h$ be the height of the cone and $\Pi$ the atmospheric pressure. When the water within the cone has risen half-way up, suppose $x$ is the depth of the vertex and $\Pi'$ the pressure of the air enclosed within the cone.



Since the volume of a cone is proportional to the cube of its height, we have by Boyle's law,

$$\Pi h^3 = \Pi'(h/2)^3,$$

or
$$\Pi' = 8\Pi. \quad . \quad . \quad . \quad (1)$$

Since the pressure at the water surface inside the cone is equal to the pressure at the same horizontal level outside the cone, we get

$$\Pi + (x + h/2)w = \Pi' = 8\Pi, \text{ from } (1),$$

or
$$(x + h/2)\, w = 7\Pi, \quad . \quad . \quad . \quad (2)$$

where $w$ is the weight of unit volume of water.

If $H$ be the height of the water barometer, then $Hw = \Pi$. Hence from (2), we get

$$(x + h/2) = 7H,$$

or
$$x = 7H - h/2.$$

(iv) *The radius of a sphere, containing air, is doubled and the temperature raised from* 0°C. *to* 455°C. *Show that the pressure of the air inside the sphere is reduced to one-third its original value, the coefficient of expansion of air per* 1°C. $= 1/273$. [*U. P. C. S.*, 1938]

Suppose $p_0$, $v_0$, $\varrho_0$, are the initial values of pressure, volume and

density respectively of the air contained in the sphere; let them be $p$, $v$, $\varrho$ when the radius is doubled.

If $\alpha$ be the coefficient of expansion for air, we have by Charles' law

$$p = K\varrho\,(1 + \alpha t),$$

or

$$pv = K\varrho v\,(1 + \alpha t)$$
$$= Km\,(1 + \alpha t), \qquad \ldots\ldots \quad (1)$$

where $m$ is the mass of the air.

$$\therefore \qquad p_0 v_0 = Km. \qquad \ldots\ldots \quad (2)$$

When the radius is doubled, the volume of the air becomes eight times its initial volume. Hence putting in (1), $\alpha = \frac{1}{273}$ and $t = 455$, we get

$$8pv_0 = Km(1 + \tfrac{455}{273}). \qquad \ldots\ldots \quad (3)$$

From (2) and (3), we have

$$\frac{8p}{p_0} = 1 + \frac{455}{273} = \frac{728}{273},$$

or

$$\frac{p}{p_0} = \frac{91}{273} = \tfrac{1}{3},$$

or

$$p = \tfrac{1}{3}p_0.$$

**Alternative Method**

The example can be done in a shorter way by employing the gas equation, namely

$$\frac{PV}{T} = \text{const.}$$

In this case we have

$$\frac{p_0 \cdot v_0}{273} = \frac{p \cdot 8v_0}{273 + 455},$$

or

$$p = \frac{728}{8 \times 273}p_0,$$

or

$$p = \tfrac{1}{3}p_0.$$

(v) 1000 *c. inches of air under a pressure* 20 *lbs. per square inch are mixed with* 800 *c. inches of air under a pressure of* 15 *lbs. per sq. inch. Find the pressure of the mixture when it has a volume of* 1500 *c. inches, the temperature being the same in all cases.*

When 1000 c. inches of air at 20 lbs. per square inch pressure is expanded to 1500 c. inches, the pressure by Boyle's law

$$= \frac{1000 \times 20}{1500} \text{ lbs. per sq. inch}$$
$$= \tfrac{40}{3} \text{ lbs. per sq. inch.}$$

Similarly, when 800 c. inches of air at 15 lbs. per sq. inch pressure is expanded to 1500 c. inches, the pressure

$$= \frac{800 \times 15}{1500} \text{ lbs. per sq. inch}$$

$$= 8 \text{ lbs. per sq. inch.}$$

Hence, the pressure required, by (2) of **6·5**,

$$= (\tfrac{40}{3} + 8) \text{ lbs. per sq. inch}$$

$$= 21\tfrac{1}{3} \text{ lbs. per sq. inch.}$$

## Examples XIV

1. Given that the sp. gr. of air at the earth's surface at a given place is 0·00125 and of mercury is 13, when the barometer is at 30″, find the height of the homogeneous atmosphere.

2. If the sp. gr. of air is 0·0013 and that of mercury 13·568, and if the height of the mercury barometer is 30 inches, prove that, in foot-second units, the value of $K$ from the formula $p=K\rho$ is 26092 gr. approximately.

3. Find the height of the homogeneous atmosphere corresponding to a barometric height of 760 mm. of mercury, taking the sp. gr. of air 0·0013 and that of mercury 13·596. [*Nagpur*, 1933]

4. The mass of a litre of air at 760 mm. pressure and 0°C. is 1·290 grammes. Find the mass of 1 cu. meter of air at a pressure of 1·9 mm.

5. At a depth of 10 feet in a pond the volume of an air bubble is 0·0001 of a cubic inch; find approx. what it will be when it reaches the surface, if the height of the barometer is 30 inches, and the sp. gr. of mercury 13·5. [*Lucknow*, 1934]

6. A right circular cylinder, open at the lower end and 9 ft. in length, is sunk into water to such a depth that the water rises $3\frac{3}{4}$ ft. inside it. At what depth is the surface of the water inside the cylinder, if the water barometer stands at 34 feet?

7. A cylindrical tube, $25\frac{1}{2}$″ in length, closed at one end, is immersed vertically in water so that the closed end is in the surface of the water. Show that the water will rise $1\frac{1}{2}$ inches in the tube, if the height of the water barometer be 32 ft.

8. If the volume of a certain quantity of air at a temperature of 10°C. be 300 cu. cm., what will be its volume (at the same pressure) when its temperature is 20°C.? [*Calcutta*, 1938]

9. At sea-level the barometer stands at 750 mm. and the temperature is 7°C., while on the top of a mountain it stands at 400 mm. and the temperature is 13°C. Compare the weights of a cubic meter of air at the two places. [*Calcutta*, 1937]

10. A litre of air at 0°C. and under atmospheric pressure weighs 1.2 grammes. Find the mass of the air required to produce at 18°C. a pressure of 3 atmospheres in a volume of 75 c. cm.

11. A mass of air at temperature 50°C. and pressure $33\frac{1}{4}$ inches of mercury, is compressed until its density is $\frac{5}{4}$ths of what it was before, its temperature at the same time falling to 16°C.; find the new pressure.

12. Some gas occupies a chamber of 80 c. cm. and has a pressure of 130 cm. of mercury at 12°C. It is then allowed to expand to a volume of 150 c.cm.; determine the temperature needed to give it a pressure of 76 cm. of mercury.

13. If a vessel containing 25 grammes of hydrogen at a pressure due to 25 inches of mercury is allowed to communicate with one containing 2·5 grammes hydrogen at a pressure of 25 ft. of water, find the pressure of the mixture when equilibrium is restored.

14. Masses $m$, $m'$ of two gases in which the ratios of the pressure to the density are respectively $k$ and $k'$ are mixed at the same temperature. Prove that the ratio of the pressure to the density in the compound is

$$\frac{mk + m'k'}{m + m'}. \qquad [M.\ T.]$$

15. The same quantities of atmospheric air are contained in two hollow spheres, their internal radii being $r,r'$ and the temperatures $t,t'$ respectively; compare the whole pressures on the surfaces.

16. A room, not hermetically sealed, contains 150 lbs. of air when the temperature is 10°C. and the pressure equal to 29 inches of mercury. What is the weight of the air in the room when the temperature falls to 0°C. and pressure rises to 30 inches of mercury?

17. A hollow cylinder of height $b$, open at the top, is inverted, and partly immersed in water with its length $a$ in air. Find the difference between the water levels outside and inside the cylinder.

18. A pipe 15 ft. long, closed at the upper end, is placed vertically in a tank of the same height; the tank is filled with water; show that, if the height of the water barometer be 33 ft. 9 inches, the water will rise 3 ft. 9 inches in the pipe.

19. A closed air-tight cylinder, of height $2a$, is half full of water and half full of air at atmospheric pressure which is equal to that of a column of height $b$ of the water. Water is introduced without letting the air escape so as to fill an additional height $k$ of the cylinder, and the pressure of the base is thereby doubled. Prove that

$$k = a + b - \sqrt{ab + b^2}.$$

20. A piston, the weight of which is equal to the atmospheric pressure on one of its ends, is placed in the middle of a hollow cylinder which it exactly fits, so as to leave a length $a$ at each end filled with atmospheric air. The ends of the cylinder are then closed and the cylinder is placed with its axis inclined at an angle $\alpha$ to the vertical, show that the piston will rest at a distance $a\{(1 + \sec^2 \alpha)^{1/2} - \sec \alpha\}$ from the former position.

21. A closed cylinder with its axis vertical is filled with two gases which are separated by a heavy piston. Determine the position of the piston, it being given that either fluid, if it filled the whole cylinder, would support a pressure equal to $\frac{3}{8}$ths of the weight of the piston.

22. A thin conical surface of weight $W$ just sinks to the surface of a fluid, when immersed with its open end downwards, but when immersed with its vertex downwards a weight equal to $mW$ must be placed within it to make it sink to the same depth as before, show that if $a$ be the length of the axis, $b$ the height of the barometric column of the fluid,

$$a/b = m(1 + m)^{1/3}.$$

23. A volume $v$ of a gas at temperature $t$ and pressure $p$ is mixed with a volume $v'$ of another gas at temperature $t'$ and pressure $p'$. If the volume of the mixture is $V$ and the temperature $T$, find the pressure. [*Patna*, 1935]

24. A piston without weight fits into a vertical cylinder, closed at its base and filled with air, and is initially at the top of the cylinder; if water be slowly poured on the top of the piston, show that the upper surface of the water will be lowest when the depth of the water is $(\sqrt{ab} - b)$, where $b$ is the height of the water barometer, and $a$ the height of the cylinder.

25. If $p_1, \rho_1, t_1, p_2, \rho_2, t_2, p_3, \rho_3, t_3$ be the corresponding values of the pressure, density and temperature of the same gas, show that $t_1(p_2/\rho_2 - p_3/\rho_3) + t_2(p_3/\rho_3 - p_1/\rho_1) + t_3(p_1/\rho_1 - p_2/\rho_2) = 0.$ [*M.T.*]

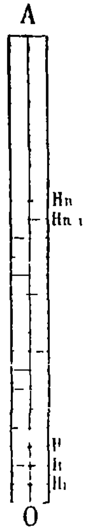
**6·7. Relation between the Altitude and the Density of the air.** As we ascend from the level of the sea, air becomes rarefied and its density changes. We establish now a law which, under certain conditions, enables us to determine the way in which density changes with the change of altitude.

*If the atmosphere be at rest and the temperature and the force of gravity be assumed to remain constant, then, as the altitude increases in Arithmetical Progression, the common difference being small, the density diminishes in Geometrical Progression.*

Take a vertical column of air of unit horizontal cross-section, whose axis is the straight line $OA$. Let $H_1, H_2, H_3, \ldots, H_n$ be a series of points on $OA$ in $A.P.$, so that

$$OH_1 = H_1H_2 = H_2H_3 = \ldots = H_{n-1}H_n = \beta,$$

where $\beta$ is small.



Now consider the vertical column of air to be divided into a number of strata of depth $\beta$ by horizontal planes passing through $H_1, H_2, \ldots$ .

Suppose the densities of the layers passing through $O, H_1, H_2, \ldots, H_{n-1}$ are $\rho_1, \rho_2, \rho_3, \ldots, \rho_n$ respectively. Since $\beta$ is small, the density for each layer may be considered uniform and equal to that at its lower surface.

Considering the equilibrium of the layer $OH_1$ of air, we get

upward pressure through the face at $O$

= downward pressure through the face at $H_1$ + wt of the layer of air,

or $$K\rho_1 = K\rho_2 + g\rho_1\beta,$$

since the pressure = $K \times$ density, $K$ being a constant.

Similarly, for the columns $H_1H_2, H_2H_3, \ldots, H_{n-1}H_n$, we have

$$K\rho_2 = K\rho_3 + g\rho_2\beta,$$
$$K\rho_3 = K\rho_4 + g\rho_3\beta,$$
$$\cdots\cdots\cdots$$
$$K\rho_{n-1} = K\rho_n + g\rho_{n-1}\beta.$$

Hence

$$\rho_2 = \rho_1(1 - \beta g/K),$$
$$\rho_3 = \rho_2(1 - \beta g/K) = \rho_1(1-\beta g/K)^2,$$
$$\rho_4 = \rho_3(1 - \beta g/K) = \rho_1(1 - \beta g/K)^3.$$
$$\cdots\cdots\cdots\cdots\cdots$$
$$\rho_n = \rho_{n-1}(1 - \beta g/K) = \rho_1(1 - \beta g/K)^{n-1}.$$

It is, therefore, clear that the densities, and consequently also the corresponding pressures, diminish in G P., as the altitudes increase in A. P.

**6·71. Density of air at a given height.** Referring to the figure of the last article if $\rho$ be the density at $H_n$, and $\rho_1$ at the starting point $O$, we have

$$\rho = \rho_n(1 - \beta g/K) = \rho_1(1 - \beta g/K)^n.$$

Putting $n\beta = Z$, we have

$$\rho = \rho_1(1 - gZ/nK)^n.$$

Now, if

$$gZ/nK = 1/x, \text{ so that } n = gZx/K,$$

then $\rho=\rho_1(1 - 1/x)^{gzx/k} = \rho_1[(1 - 1/x)^{-x}]^{-gz/k}$

Now keeping $Z$ constant, let $n$ tend to infinity so that $x$ also will tend to infinity, and we get in the limit

$$\rho = \rho_1 \cdot e^{-gz/k}, \quad \ldots \quad (1)$$

which gives the density of air at a height $Z$ in terms of the density at the point from which the height is measured.

If the starting point be the surface of the earth, then (1) gives the density at a height $h$ in terms of the density at the surface of the earth.

It must be noted that the above formula has been obtained on the assumptions that the temperature and $g$ are both constant throughout. But since neither of these two assumptions is even approximately true for large differences of altitudes, the formula can give an approximately correct result only for small variations in altitudes.

**Alternative Method**

The formula (1) can be obtained in a simpler manner by the use of *Integral Calculus*; the proof is given below.

Taking a vertical column of air as in **6·7**, let $p$ be the pressure at a height $z$, $p + \Delta p$ at height $z + \Delta z$, where $\Delta z$ is small, and $\varrho$ the density at height $z$. We have by (4) of **6·33**

$$p = K\varrho, \qquad \ldots\ldots \quad (2)$$

where $K$ is a constant.

Now since the thin column $\Delta z$ of air is being pressed upwards with pressure $p$ and downwards with $p + \Delta p$, we get by considering the equilibrium of this column of air

$$p = p + \Delta p + g\varrho\Delta z.$$

Taking the limit, we get

$$dp/dz = -g\varrho.$$

Making use of (2), we have

$$d\varrho/dz = -\varrho g/K,$$

or
$$\int \frac{d\varrho}{\varrho} = \int -\frac{g}{K}\, dz,$$

or
$$\log \varrho = -zg/K + \text{a constant } C. \quad (3)$$

When $z = 0$, let $\varrho = \varrho_1$. Then

$$\log \varrho_1 = -g/K \times 0 + C. \quad \ldots \quad (4)$$

Subtracting (4) from (3), we get

$$\log(\varrho/\varrho_1) = -zg/K,$$

or
$$\varrho = \varrho_1\, e^{-zg/k}. \qquad \ldots \quad (5)$$

If $p$ and $p_1$ be the pressures corresponding to densities $\varrho$ and $\varrho_1$ respectively, then we have

$$p = p_1\, e^{-zg/k}. \qquad \ldots\ldots \quad (6)$$

**6·72. Determination of heights by Barometer.** Let $h$, $h_1$ be the barometric readings at the upper and lower stations, $\varrho$ and $\varrho_1$ the densities of air at these places and $z$

the difference of their altitudes. Then, if $p$ and $p_1$ denote the atmospheric pressures at those places, we have

$$h/h_1 = p/p_1 = \varrho/\varrho_1.$$

But from (5) of 6·71, we have

$$\varrho/\varrho_1 = e^{-gz/k},$$

$$\therefore \qquad e^{-gz/k} = h/h_1,$$

$$\text{or} \qquad -zg/K = \log (h/h_1),$$

$$\text{or} \qquad Z = \frac{K}{g} \cdot \log \frac{h_1}{h}, \qquad \ldots \quad (1)$$

*which gives the height between the two stations in terms of the barometric readings at those places.*

In particular, if $h_1$ be the barometric reading at the surface of the earth and $h$ the reading at the place whose height is to be determined, then the required height $z$ is given by (1).

It was Pascal who for the first time had suggested the valuable idea that in order to form an estimation of the height of a mountain the barometric readings at the foot and at the top of the mountain should be compared. This idea was experimented upon in 1648 by his friend Perier who ascended Puy de Dome in Auvergne (France) and observed that the barometric readings had fallen at the top of the mountain by nearly 4 inches.

**6·8. Faulty Barometers.** We have seen that in order that a barometer be a correct one, the space above the mercury in the long tube must be a vacuum. If this space is not a perfect vacuum, the barometer is defective and the readings given by it will not be correct. Such barometers are usually called *faulty barometers*, and we shall illustrate by means of examples how to deal with faulty barometers.

**6·9. Illustrative Examples.** (1) *A barometer which has a little air in it reads* 29.6 *inches, the end of the tube being* 6 *inches above the top of mercury, when the true pressure of the atmosphere is* 30 *inches. What is the reading of the barometer when the true pressure is* 29 *inches?*

The length of the tube

$$= (29 \cdot 6 + 6) \text{ inches} = 35 \cdot 6 \text{ inches.}$$

The pressure of the air which occupies a length of 6 inches above mercury is that due to $0 \cdot 4 \ (= 30 - 29 \cdot 6)$ inch of mercury.

If the reading of the barometer in the second case be $x$ inches, then the length occupied by the air this time is $(35 \cdot 6 - x)$ inches and its pressure is that due to $(29 - x)$ inches of mercury.

Now applying Boyle's law, we have

$$(35 \cdot 6 - x)(29 - x) = 6 \times 0 \cdot 4,$$

or $$x^2 - 64 \cdot 6x + 1030 = 0,$$

or $$(x - 32 \cdot 3)^2 = 13 \cdot 29,$$

or $$x - 32 \cdot 3 = \pm 3 \cdot 65 \text{ approx.},$$

or $$x = 28 \cdot 65,\ 35 \cdot 95.$$

The value $35 \cdot 95$ being obviously inadmissible, the required reading is $28 \cdot 65$ inches.

(ii) *A barometer with an imperfect vacuum stands at* 29·8 *and* 29·4 *inches when a correct barometer indicates* 30·4 *and* 29·8 *inches respectively. When the faulty barometer stands at* 29 *inches, what will be the reading of the correct barometer?* [*Bombay*, 1936]

Let $x$ be the length in inches of the column of mercury which equals the pressure of the air enclosed in the vacuum of the barometer in the third case. We have then the following data:—

| | | | |
|---|---|---|---|
| Reading in faulty barometer— | 29·8″ | 29·4″ | 29″ |
| Reading in correct barometer— | 30·4″ | 29·8″ | reqd. |
| Pressure of the air enclosed— | 0·6″ | 0·4″ | $x''$ |

If the length of the tube be $l$ inches, the lengths occupied by the enclosed air in the three cases are respectively $l - 29 \cdot 8$, $l - 29 \cdot 4$ and $l - 29$ inches.

By Boyle's law, we have

$$(l - 29 \cdot 8) \times 0 \cdot 6 = (l - 29 \cdot 4) \times 0 \cdot 4 = (l - 29) \times x.$$

From the first two of the above equations, we get

$$2l = 178 \cdot 8 - 117 \cdot 6,$$

or $$l = 30 \cdot 6 \text{ inches.}$$

Again, substituting for $l$, we have

$$(30 \cdot 6 - 29 \cdot 4) \times 0 \cdot 4 = (30 \cdot 6 - 29) \times x,$$

or $$16x = 4 \cdot 8,$$

or $$x = 0 \cdot 3.$$

Hence, the required correct reading is 29·3 inches.

(iii) *The readings of a perfect mercurial barometer are $\alpha$ and $\beta$ when the corresponding readings of a faulty barometer are $a$ and $b$. Prove that the correction to be applied to any reading $c$ of the faulty barometer is*

$$\frac{(\alpha - a)(\beta - b)(a - b)}{(\alpha - a)(a - c) - (\beta - b)(b - c)}.$$

[*U. P. C. S.*, 1941; *Allahabad*, 1934, 1937]

Let $x$ be the correction required, and $l$ the length of the tube. By Boyle's law, we have

$$(l - a)(\alpha - a) = (l - b)(\beta - b) = (l - c)x.$$

From the first two, we get

$$l(\alpha - \beta - a + b) = a(\alpha - a) - b(\beta - b).$$

$$\therefore \quad l = \frac{a(\alpha - a) - b(\beta - b)}{\alpha - \beta - a + b}.$$

Again

$$x = \frac{(l - b)(\beta - b)}{(l - c)}. \quad \ldots \quad (1)$$

Now

$$l - b = \frac{\alpha(a - b) - a(a - b)}{(\alpha - a) - (\beta - b)} = \frac{(\alpha - a)(a - b)}{(\alpha - a) - (\beta - b)};$$

$$l - c = \frac{(\alpha - a)(a - c) - (\beta - b)(b - c)}{(\alpha - a) - (\beta - b)}.$$

Putting the values of $l - b$ and $l - c$ in (1), we get

$$x = \frac{(\alpha - a)(\beta - b)(a - b)}{(\alpha - a)(a - c) - (\beta - b)(b - c)}.$$

(iv) *If a change from 30 inches to 27 inches in the barometric height correspond to a rise in the altitude of 2290 ft., find the altitude which corresponds to the barometric height of 24 inches.*

(log 2 = 0·3010, log 3 = 0·4771.)

[*Bombay*, 1935; *Allahabad*, 1932, 1940]

Making use of the formula (6) of **6·71**, we get

$$30 = p_1 e^{-gh/K},$$

where $h$ is the height of the first station and $p_1$ the pressure at the point from which the height is measured. Again

$$27 = p_1 e^{-g(h + 2290)/K},$$

$$24 = p_1 e^{-g(h + H)/K},$$

where $H$ is the height of the third station above the first.

We have

$$\tfrac{30}{27} = e^{2290\, g/K}, \qquad \ldots\ldots\ldots \quad (1)$$

$$\tfrac{30}{24} = e^{Hg/K}. \qquad \ldots\ldots\ldots \quad (2)$$

From (1), we get

$$2290\, g/k = \log_e \tfrac{10}{9} = \log_{10} \tfrac{10}{9} \times \log_e 10.$$

From (2), we have

$$\frac{Hg}{k} = \log_e \tfrac{5}{4} = \log_{10} \tfrac{5}{4} \times \log_e 10.$$

$$\therefore \qquad \frac{H}{2290} = \frac{\log_{10} \frac{5}{4}}{\log_{10} \frac{10}{9}} = \frac{\log 10 - 3 \log 2}{\log 10 - 2 \log 3}$$

$$= \frac{1 - \cdot 9030}{1 - \cdot 9542} = \frac{0 \cdot 0970}{0 \cdot 0458} = \frac{970}{458},$$

or
$$H = \frac{2290 \times 970}{458} \text{ ft.} = 5 \times 970 \text{ ft.}$$

$$= 4850 \text{ ft.}$$

(v) *A station B is at an altitude h above another station A. If* $P_B$, $P_A$ *are the pressures at B and A, prove that*

$$P_B = P_A\, e^{-h/H},$$

*where H is a constant.*

*A hollow gas-tight balloon, containing helium, weighs W lbs. When its lowest point touches the ground, it requires a force of w lbs. to prevent it from rising. Show that it can float in equilibrium at a height*

$$H \log_e (1 + w/W). \qquad \text{[}Allahabad\text{, 1924]}$$

For the first part of the question, see the proof of the formula (6) of **6·71.**

If $V$ be the volume of the balloon and $\varrho_0$ the density of the air on the ground, then

$$w + W = g\varrho_0 V. \qquad \ldots\ldots \quad (1)$$

If it floats in equilibrium at a height $h$ above the ground, where the density of air is $\varrho$, we have

$$W = g\varrho V. \qquad \ldots\ldots\ldots \quad (2)$$

Also if $P_A$ and $P_B$ denote pressures at the surface of the earth and at height $h$ respectively, then

$$P_B = P_A\, e^{-h/H},$$

or
$$\varrho = \varrho_0\, e^{-h/H}. \qquad \ldots\ldots \quad (3)$$

From (1) and (2) we get

$$\rho_0/\rho = 1 + w/W,$$

or $$e^{h/H} = 1 + w/W \text{ from (3)},$$

or $$h = H \log_e (1 + w/W).$$

## Examples XV

1. A barometer stands at 30 inches, the vacuum above the mercury being perfect, the area of the cross-section of the tube is 0·2 sq. inch. If 0·2 cubic inch of ordinary air be introduced into the vacuum, the mercury is seen to fall through 3 inches. Find the length of the original vacuum. [*Lucknow*, 1942]

2. 10 c.cm. of air, measured at atmospheric pressure, when introduced into a barometer vacuum depress the mercury which previously stood at 76 cm and occupy a volume of 15 c.cm. By how much has the mercurial column been depressed?

3. A bubble of air having a volume of one cubic inch at a pressure of 30 inches of mercury escapes up a barometer tube whose cross-section is one square inch and whose vacuum is one inch long. How much will the mercury descend? [*Benares*, 1943]

4. A faulty barometer tube 33 inches long contains air at the top and consequently reads 28 inches when the true pressure of the atmosphere is 29. Find its reading when the pressure of the atmosphere is 28 inches.

5. A barometer which is known to have some air above the mercury is constituted so that the tube can be depressed into the cistern, thus varying the volume of the tube above the mercury column. When the top of the column is 5 inches below the top of the tube, the barometer reads 30 inches. On depressing the tube by 3 inches the barometer reads 29·5 inches. Find what would be the correct reading when there be no air above the mercury.

6. The reading of a faulty barometer, the tube of which contains a quantity of air of length 3½ inches, is 28 inches, when the reading of a true barometer is 30 inches. What will be the reading of the faulty barometer when the true barometer reads 29 inches? [*Agra*, 1929]

7. The height of the Torricellian vacuum in a barometer being 3 inches, the instrument indicates a pressure of 29 inches when the true barometer reads 30 inches. Assuming that the faulty readings are due

to the pressure of some air in the vacuum, show that the true reading corresponding to any faulty reading $h$, is $h + 3/(32 - h)$.

8. The height of the Torricellian vacuum in a barometer is $a$ inches, and the instrument indicates a pressure of $b$ inches of mercury when the true reading is $c$ inches. The faulty readings are due to an imperfect vacuum. Prove that the true reading corresponding to an apparent reading of $d$ inches is

$$d + \frac{a(c - b)}{a + b - d}.$$ [*Allahabad*, 1941; *Lucknow*, 1928, 1937]

9. A bent uniform tube has two equal vertical branches close together, one end being open and the other end closed. Mercury is poured into the open end and no air escapes. If when the mercury just fills the open tube, the air occupies two-thirds of the closed branch, prove that the length of either branch is equal to three times the height of the mercurial barometer.

10. If a change from 30″ to 28″ in the height of the barometer corresponds to a change of altitude of 1800 ft., find the change of altitude corresponding to a change of barometric height from 30″ to 26·4″.

$$[\log 2 = 0{\cdot}3010, \log 3 = 0{\cdot}4771, \log 7 = 0{\cdot}8451, \log 11 = 1{\cdot}0414.]$$

11. The density of mercury is 13·6 and that of air at 760 mm. pressure 0·001293. Prove that when the reading of the mercury barometer is 750 mm., its reading at an attitude 500 metres is about 704·7 mm., the variation of temperature being neglected.

[$\log 2 = 0{\cdot}3010$, $\log 3 = 0{\cdot}4771$, $\log_e 10 = 2{\cdot}3026$, $\log 70{\cdot}47 = 1{\cdot}848$.] [*Bombay*, 1937]

12. Assuming that a change from 30″ to 27 inches in the height of the barometer corresponds to an altitude of 2700 ft., find the altitude corresponding to the height 21·87″ of the barometer.

($\log 3 = 0{\cdot}4771$.) [*Benares*, 1934]

13. A cylindrical well of depth $h$ and section $A$ is maintained at constant temperature. If $\varrho_0$ and $\varrho_1$ are the densities of the air at the top and bottom, show that the total amount of air contained is

$$\frac{Ah(\varrho_1 - \varrho_0)}{\log \varrho_1 - \log \varrho_0}.$$ [*Bombay*, 1936]

14. A heavy gas at constant temperature is confined in a vertical

cylinder of height $h$. If $\rho_0$ be the density at the base, prove that the mean density is

$$\frac{k\rho_0}{gh}(1 - e^{-g\,h/k}).$$

15. A box is filled with a heavy gas at uniform temperature. Prove that if $a$ is the altitude of the highest point above the lowest, and $p$ and $p'$ are the pressures at these points, the ratio of the pressure to the density at any point is equal to

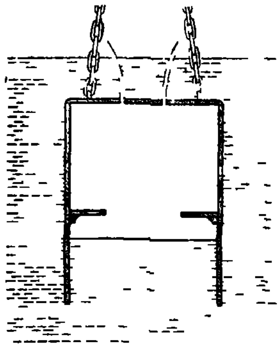
$$\frac{ag}{\log (p'/p)}.$$ [*Nagpur*, 1931]

CHAPTER VII

# MACHINES DEPENDING UPON FLUID PROPERTIES

**7·1. Utilization of fluid properties.** In previous chapters we have obtained a number of properties of liquids and gases. We have seen in the case of Bramah's press [1·25] how the principle of the transmissibility of pressure is utilized in producing enormously big forces. There is a large number of machines and instruments *which depend for their working on the properties of fluids*. In order to illustrate the application of these properties we select some simple machines and explain the principles on which they work. For detailed mechanical descriptions, reference may be made to some suitable technical book on the subject.

**7·2. The Diving Bell.** This machine made of metal, is a hollow, nearly cylindrical, or bell-shaped, vessel closed at the top and open at its lower end. It is sufficiently heavy to sink under its own weight along with the air enclosed in it. Its object is to enable divers to go to the bottom of deep waters in order to lay the foundation of piers or to do such other operations as are required



there. It is usually capable of providing seating accommodation for several persons on platforms, as shown in the

figure. The machine is lowered into water by means of a chain attached to its top.

When the bell goes down into water, the pressure of the enclosed air increases, because the pressure of this air is always the same as that at the level of the water with which it is in contact. In virtue of Boyle's law, the volume of the enclosed air decreases and the water rises in the bell to a height which increases with the depth of the bell from the surface of the water.

The machine is provided with a tube connecting the interior of the bell with the atmospheric air. Through this tube fresh air can be pumped into the bell to ensure a sufficient supply of oxygen and to adjust the amount of air in order to keep the water at any desired level inside the bell. There is another tube through which the air from the inside of the bell can be taken out.

Considering the equilibrium of the machine we find that the tension of the chain is equal to the difference of the weight of the bell together with the air enclosed and the weight of the water displaced by it. As the bell descends, if no additional air from outside be pumped in, the volume of the enclosed air becomes less and less and therefore the volume of the water displaced by the bell steadily decreases. Consequently the tension of the chain becomes greater and greater as the bell goes down lower and lower.

**7·21. Problems on the Diving Bell.** *A cylindrical diving bell of height $h$ is lowered into water till its top is at depth $a$ below the water surface. If the temperature is assumed to remain constant to find*

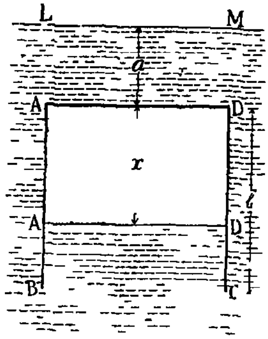
(i) *the rise of water in the bell,*

(ii) *the tension of the chain at this depth, and*

(iii) *the volume of air at atmospheric pressure that must*

*be forced in from above so that no water may remain within the bell.*

(i) When the top $AD$ of the bell $ABCD$ is at depth $a$ from the water surface $LM$, suppose the water inside the bell has risen to the level $A'D'$. Then the air which originally occupied the space $ABCD$ is compressed to $AA'D'D$. Let $AA' = x$, and $K$ the internal cross-section of the bell.




If $h$ be the height of the water barometer, $w$ the weight of unit volume of water and $\Pi$ the atmospheric pressure, then

$$\Pi = wh. \qquad . \quad . \quad . \quad . \quad (1)$$

The pressure of the compressed volume $AA'DD'$ of air, being the same as the pressure at the water level $A'D'$, is $w\,(x + a) + \Pi$, or $w(x + a + h)$. The original volume $ABCD$ of air was at atmospheric pressure $\Pi$. Hence applying Boyle's law, we get

$$wh \times bK = w\,(x + a + h) \times xK,$$

or
$$x^2 + (a + h)x - hb = 0, \qquad . \quad . \quad . \quad (2)$$

which is a quadratic equation in $x$ having one positive and one negative root. Taking the positive root, the required rise of water in the bell will be given by $b - x$.

(ii) Let $T$ denote the tension of the chain which supports the bell and $W$ the weight of the bell. The tension is equal to the weight of the bell together with the air enclosed, less the weight of the water displaced. Hence, neglecting the weight of the enclosed air which is very small, we have

$$T = W - wKx. \qquad . \quad . \quad . \quad (3)$$

(iii) Let $V$ be the volume of the original air at atmospheric pressure $\Pi$ contained in the bell, and suppose $V'$ is the volume of the air at atmospheric pressure that must be forced in so that the water level may be kept at $BC$. The pressure of the air in this case, being the same as the pressure at the water level $BC$, will be $w(a+b)+\Pi$, or $w(a+b+h)$.

Since a total volume $(V+V')$ of air at pressure $\Pi(=wh)$ has been compressed to volume $V$ at pressure $w(a+b+h)$, it follows by Boyle's law that

$$(V+V').wh = V.w(a+b+h),$$

or
$$V' = \frac{a+b}{h}\cdot V, \quad . \quad . \quad . \quad (4)$$

which gives the required volume of the air that must be forced in to keep the bell free from water.

Let us suppose now that the bell is made to descend with a uniform velocity $v$, that is, $a$ is increasing at a uniform rate $v$. Then it follows by differentiating the above equation (4) with regard to time, that the rate at which the atmospheric air must be pumped in to keep the bell free from water, is

$$v\cdot\frac{V}{h}. \quad . \quad . \quad . \quad . \quad (5)$$

**7·22. Illustrative Examples.** (1) *A cylindrical diving bell, whose height is* 6 *feet, is let down till its top is at a depth of* 80 *ft. Find the pressure of the contained air, the height of the water barometer being* $33\frac{1}{3}$ *ft.*

[*Patna*, 1941]

If $a$ be the depth of the top of the bell below the water, $b$ its height, $h$ the height of the water barometer and $x$ the length of the bell occupied by the compressed air, then we have from **7·21** (2),

$$x^2+(a+h)x-bh=0.$$

Putting the values of $a$, $b$ and $h$ in the above, we get

$$x^2+(80+100/3)x-6\times 100/3=0,$$

or
$$3x^2+340x-600=0,$$

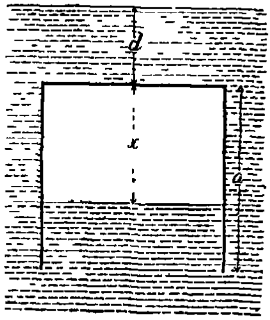
or
$$x=\tfrac{1}{3}\{-170+\sqrt{(170)^2+1800}\}$$
$$=5/3 \text{ ft. nearly.}$$

Now, if $w$ be the weight of unit volume of water, the pressure of the contained air

$$= (80 + 5/3 + 100/3)w$$
$$= (345/3)w$$
$$= 3 \cdot 45 \text{ atmospheres (nearly).}$$

(ii) *A cylindrical diving bell of height $a$ is furnished with a barometer and lowered into a fluid, the height of the mercury in the barometer before and after immersion being $h$, $h'$ respectively. Show that the depth of the bottom of the bell below the surface of the fluid is equal to $(\theta/\beta + a/h')(h' - h)$, where $\theta$ is the sp. gr. of mercury and $\beta$ that of the fluid.* [*Allahabad*, 1933]

Let $d$ be the depth of the top of the bell and $x$ be the length occupied by air. Then




$$h'\theta = h\theta + (d + x)\beta.$$

By Boyle's law

$$h'x = ha.$$

$$\therefore \quad h'\theta = h\theta + (d + ha/h')\beta,$$

or $\quad (h' - h) \,.\, \theta/\beta = d + ah/h'$,

or
$$d + a = (h' - h)\theta/\beta + a - ah/h'$$
$$= (h' - h)\,\theta/\beta + (h' - h)\,a/h'$$
$$= (h' - h)(\theta/\beta + a/h').$$

(iii) *If a cylindrical diving bell of height $a$ whose chamber could contain a weight $W$ of water, be lowered so that the depth of the highest point is $d$, prove that when the temperature is raised $t°$, the tension of the supporting chain is diminished by*

$$\frac{Wht}{T\sqrt{\{(h+d)^2 + 4ah\}}} \textit{ nearly,}$$

*$h$ being the height of the water barometer and $T$ the absolute temperature.*

[*M. T.*]

If $x$ be the length occupied by the compressed air at temperature $T$, then we have from **7·21** (2),

$$x^2 + x(h + d) - ah = 0. \qquad \ldots\ldots\ldots \quad (1)$$

In the second case when the temperature has been raised by $t°$, suppose $y$ is the length occupied by the compressed air. Let $A$ denote the area of the cross-section of the bell.

Applying the formula $PV/T = K$ of **6·43** (2), we get

$$Ax(h + d + x) = KT,$$
$$Ay(h + d + y) = K(T + t).$$

$$\therefore \qquad \frac{y(b+d+y)}{x(b+d+x)} = \frac{T+t}{T} = 1 + t/T,$$

or
$$(y^2 - x^2) + (b+d)(y-x) = x(b+d+x).t/T$$
$$= ab\, t/T, \text{ from (1)}.$$

$$\therefore \qquad y - x = \frac{abt}{T} \times \frac{1}{(y+x+b+d)}$$
$$= \frac{abt}{T(2x+b+d)} \text{ nearly}, \quad . \quad . \quad . \quad (2)$$

as $y$ is supposed to differ very slightly from $x$.

Now, if $W'$ be the weight of the bell and $T_1$ and $T_2$ the tensions in the two cases, we have

$$T_1 = W' - Axw,$$
$$T_2 = W' - Ayw.$$
$$\therefore \qquad T_2 - T_1 = -Aw(y-x).$$

But since $W = Awa$, we get

$$T_2 - T_1 = -(y-x)\; W/a$$
$$= -\frac{W}{a} \times \frac{abt}{T} \times \frac{1}{(2x+b+d)} \text{ from (2)}$$
$$= \frac{Wbt}{T} \times \frac{1}{(2x+b+d)}.$$

From (1), we have

$$2x + b + d = \sqrt{\{(x+b)^2 + 4ab\}}.$$
$$\therefore \qquad T_2 - T_1 = -\frac{Wbt}{T} \times \frac{1}{\sqrt{\{(b+d)^2 + 4ab\}}}.$$

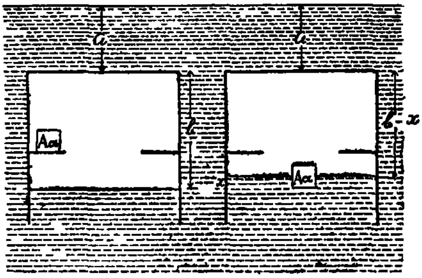
Hence, the tension of the supporting chain is diminished by

$$\frac{Wbt}{T} \cdot \frac{1}{\sqrt{\{(b+d)^2 + 4ab\}}} \text{ nearly}.$$

(iv) *A cylindrical diving bell whose cross-section is of area $A$, is suspended in water with its flat top at a fixed distance $a$ below the surface, and the enclosed air then occupies a length $b$ of the bell. A body of volume $A\alpha$ and sp. gr. $\sigma$ falls off from the platform and floats in the enclosed water. Show that the level of the water inside the bell rises, but the amount of water in the bell and the tension of the supporting chain are less than before.*

Suppose $w$ is the weight of unit volume of the water and $h$ the height of the water barometer.

Before the body falls off, the volume of the air in the bell is $A(b-a)$ at pressure $w(a+b+h)$.




When the body floats in the water, let $b-x$ be the distance of the water-level inside the bell from the top. The volume of the water displaced by the body being $Aa\sigma$, the volume $(Aa - Aa\sigma)$ of the body will be above water. Hence the volume of the air in the bell is now

$$A(b-x) - (Aa - Aa\sigma)\,,\quad \text{i.e., } A(b-x-a+a\sigma),$$

and its pressure is $w(a+b-x+h)$.

Hence, by Boyle's law

$$A(b-a)\times(a+b+h)w = A(b-x-a+a\sigma)\times(a+b-x+h)w,$$

or $$x^2 - (2b+a+h-a+a\sigma)x + a\sigma(b+a+h) = 0. \quad . \quad (1)$$

The above quadratic equation in $x$ has its second term negative (since $a < b$), and third term positive; hence its roots are positive.

Thus $x$ is positive, which means that the level of water inside the bell rises.

Now,

the initial volume of air in the bell — the final volume of air

$$= A(b-a) - A(b-x-a+a\sigma)$$
$$= A(x-a\sigma). \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Writing (1) as

$$(x-a\sigma)(x-2b-a-h+a) = a\sigma(b-a),$$

or $$(x-a\sigma)[-(b-x)-a-h-(b-a)] = a\sigma(b-a), \quad . \ (3)$$

it may be observed that the expression in the square brackets on the left-hand side in (3) is negative, and the right-hand side is positive; therefore $x - a\sigma$ is negative.

It follows from (2) that the volume of air in the bell is increased. Consequently the amount of water in the bell is decreased.

Finally, if $W$ be the weight of the bell in water, we have

the initial tension of the chain

= wt. of the bell + wt. of the body — wt. of the water displaced

$$= W + Aa\sigma w - Abw. \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Again, the tension of the chain in the second case

= wt. of the bell — wt. of the water displaced

$$= W - A(b-x+a\sigma)w. \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Hence, subtracting (5) from (4), we find that the difference between the initial and final tensions

$$= Aw(2a\sigma - x), \qquad . \quad . \quad . \quad (6)$$

which is positive, since $x - a\sigma$ has been shown above to be negative.

The tension of the supporting chain is, therefore, decreased.

## Examples XVI

1. A barometer in a cylindrical diving bell indicates a pressure of 45 inches of mercury, the height of the barometer at the surface of the earth being 30 inches. What will be the depth of the level of the water inside the bell if the sp. gr. of mercury be 13·6?

2. A cylindrical diving bell 6 feet in height and 5 feet in diameter is lowered till its top is 45 feet below the surface. What volume of air at the atmospheric pressure—that due to 34 feet of water—must be pumped in to fill the bell completely?

3. A cylindrical diving bell 10 feet in height is sunk under the sea until the water rises half-way up the bell; find the depth of the top of the bell, taking the height of water barometer as 33 feet and the sp. gr. of sea-water 1·026.

4. In the above question find the volume of the air at atmospheric pressure that must be pumped in to keep the bell clear of the water.

5. A cylindrical diving bell 8 feet in length is lowered in water till the air occupies $\frac{1}{4}$th of its volume. Find the depth of the top of the bell below the surface of the water, taking the height of the water barometer to be 34 feet. Find also the volume of air at the atmospheric pressure to be pumped in to keep the bell free from the water at this stage.

6. A cylindrical bell 4 feet long whose volume is 20 cubic feet is lowered into water until its top is 14 feet below the surface of the water, and air is forced into it until it is $\frac{3}{4}$th full. What volume would the entire quantity of air occupy under atmospheric pressure, the water barometer standing at 33 ft.? [*Calcutta*, 1912]

7. If a diving bell in the shape of an inverted cone, of height $a$, be lowered till the vertex is at a depth $d$, prove that the height $x$ of the part of the bell occupied by the air is given by the equation

$$x^4 + x^3(b + d) = a^3 b,$$

where $b$ is the height of the water barometer. [*Allahabad*, 1932]

8. A diving bell in the shape of a cone of height 12 feet is lowered in water until the water occupies $\frac{1}{3}$rd of the height of the bell.

Find the depth of the vertex below the surface of the water, if the height of the water barometer be 30 feet.

9. A circular cone, hollow, but of great weight is lowered into the sea by means of a rope attatched to the vertex. If $h$ be the height of the cone, $c$ the depth of the vertex below the free surface, $k$ the height of the water barometer, and $T$, $T'$ the absolute temperature of the air at the surface and of the water, prove that the depth of the water surface below the vertex of the cone is given by

$$x^4 + x^3(k+c) - kh^3T'/T = 0. \qquad \text{[\textit{Bombay}, 1935]}$$

10. A diving bell is made of substance whose sp. gr. is 4 and its interior will contain a quantity of water whose weight is twice that of the bell; if the bell be lowered in water till the tension in the rope is half the weight of the bell, prove that the density of air within it will be 0·8 times that of the atmosphere. [*Nagpur*, 1939]

11. A hemispherical diving bell of radius $r$ is lowered in water with its base horizontal till the water rises up to the middle point of the vertical radius of the bell. Show that the depth of the base of the bell below the surface is $(11/5)h + \frac{1}{2}r$, $h$ being the height of the water barometer.

12. A cylindrical diving bell of height $a$, is lowered till its top is at a depth $h$ below the surface of the water. If the bell be now half-full of water and air be pumped in till all the water is expelled, prove that the bell must be lowered a further distance $4H-2h$, before the bell is again half-full of water, $H$ being the height of the water barometer. [*Nagpur*, 1938]

13. The height of a cylindrical bell is $a$ feet; at the surface a mercury barometer reads $h$ feet, when the bell is lowered it reads $h'$ feet. If $\rho$ be the sp. gr. of mercury, show that the depth to which the bell has sunk is $\rho(h'-h) - ah/h'$ feet.

14. If a cylindrical diving bell of height $h$ feet contains a mercurial barometer the column of which stands at $p_0$ inches when the bell is above the surface of the water and at a height $p$ when below; show that the depth of the top of the bell below the surface

$$= \frac{13\cdot596}{12}(p-p_0) - h\left(\frac{T'p_0}{Tp}\right) \text{feet},$$

where $T$, $T'$ are the absolute temperature and 13·596 the sp. gr. of mercury.

15. In the previous question if the bell be conical, show that

the corresponding depth is

$$\frac{13\cdot 596}{12}(p-p_0)-h\left(\frac{T'p_0}{Tp}\right)^{1/3}\text{ feet.}$$

16. A diving bell is lowered into water at a uniform rate, and air is supplied to it by a force pump, so as to keep the bell full without allowing any air to escape. How must the quantity, i.e., mass of air, supplied per second vary as the bell descends? [*Agra*, 1930]

17. If a cylindrical diving bell, whose capacity is $V$ cubic feet, be sunk to such a depth that the water stands at $1/m$ th of its height, and be then lowered at a uniform rate of $n$ feet per second, prove that the number of cubic feet of air which must be pumped in per second in order that the water may always remain at the same height, will be

$$(1 - 1/m)\, nV/h,$$

where $h$ is the height of the water barometer.

18. If a diving bell descends from the surface with uniform velocity $V$, show that the water will ascend a height $h$ in the bell in time

$$\frac{h}{V}\left(1+\frac{H}{l-h}\right),$$

where $l$ is the length of the bell and $H$ the height of the water barometer.

19. A cylindrical diving bell is lowered to a given depth in water by means of a chain and is completely immersed. If it be lowered to the same depth in a lighter liquid, will the tension of the chain be greater or smaller? [*Bombay*, 1935]

20. A diving bell is in the form of a cylinder with a hemispherical top, $c$ is the length and $a$ the radius of the cylinder. Find how far the bell must be lowered in order that the hemisphere may be the only part containing air; show that in this position the volume of air at atmospheric pressure which must be forced in, to clear the bell from water, must be $(c/H + \frac{3}{2}\, c/a)$ times the volume of the bell, $H$ being the height of the water barometer.

21. A diving bell in the form of a cylinder of length $a$ is surmounted by a cone of height $b$. If no air is pumped in when it is immersed, find how far it must be lowered for all the air to be forced into the conical part. Show that the volume of air at atmospheric pressure which must now be pumped in so that the bell may be filled, is $(a/H + 3a/b)V$, where $H$ is the height of the water barometer, and $V$ the volume of the bell.

22. A cylindrical diving bell of height 10 feet and internal radius 3 feet is immersed in water so that the depth of the top is 100 feet. Show that if the temperature of the air in the bell be now lowered from 20°C. to 15°C., and if 30 feet be the height of the water barometer at this time, the tension of the chain is increased by about 67 lbs. [*M. T.*]

23. In a diving bell a soda water bottle is opened, which in the external air would liberate a volume $V$ of gas, show that the tension of the rope is diminished by

$$\frac{whV}{\sqrt{(a+b)^2+4hb}},$$

where squares of $V/hA$ are neglected, $w$ being the weight of a unit volume of water, $A$ the cross-section, $a$ the depth and $b$ the height of the bell, and $h$ the height of the water barometer in atmospheric air. [*Benares*, 1935]

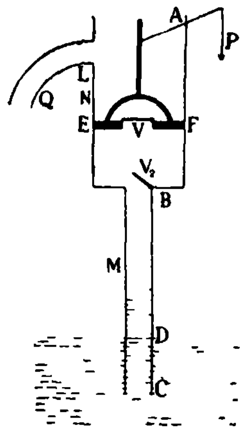
## Pumps for Liquids

7·3. All pumps depend for their working on the principle of suction. A partial vacuum is created in the space inside the pump and the pressure of the internal air thereby decreased. The external atmospheric pressure being greater than the pressure of the internal air, forces the liquid to rise up.

Valves are used in all pumps. A valve is like a trapdoor which opens in one direction only. It will thus open if there is an excess of pressure from one direction and will allow the passage of fluid, but if there is an excess of pressure from the other direction, the valve will close and stop the flow of the fluid.

Valves are generally made of metal, leather or oiled silk. Theoretically a valve should work whenever there is the slightest excess of pressure on one side. In practice, however, a definite amount of excess of pressure is required before the valve can work, and there is always some leakage.

**7·31. The Common or Suction Pump.** This is an instrument for raising water from a well or reservoir. It consists of two cylinders $AB$ and $BC$, usually called the pump-barrel and the suction-tube respectively. The upper cylinder $AB$ is of wider cross-section than the tube-section $BC$ which is long and narrow and terminates beneath the surface of water which is to be raised.




In the upper barrel there is a tightly fitting piston $EF$ which can be raised or lowered by means of a vertical rod attached to it, and the rod is worked by a lever $P$. The piston $EF$ is fitted with a valve $V_1$ which opens only *upwards*. The piston can move up to $L$ where the spout $LQ$ of the pump is situated. At the bottom of the pump-barrel there is another valve $V_2$ which also opens *upwards* and covers the orifice of the suction-tube $BC$.

**7·311. Action of the Pump.** Suppose that the pump starts working when the piston is in its lowest position at $B$ and that water has not risen in the tube $DB$. When the piston is raised, the air between the piston and the valve $V_2$ becomes rarefied and its pressure becomes less than that of the air contained in $BD$. The valve $V_2$, therefore, opens upwards and some air from $BD$ goes into the upper barrel. The pressure of the air in $BD$ having thus fallen below the atmospheric pressure, some water is raised up in the tube $DB$ above the water surface $D$.

When the piston has reached $L$, the direction of its motion is reversed. Now the air between the piston and $V_2$ becomes compressed and the valve $V_2$ is, therefore,

shut. When the air is sufficiently compressed so that its pressure exceeds the atmospheric pressure, the valve $V_1$ is raised and air begins to escape out from below the piston. The air continues escaping during the further downward motion of the piston. When the piston has returned to $B$, its first *complete stroke* is said to be finished. The valve $V_1$ of the piston is also closed.

Another upstroke of the piston follows and the water rises higher in the tube $DB$. After a few strokes the water rises into the upper barrel, provided the height of $DB$ is less than that of the water barometer. Then during each downward stroke some water passes up the piston through its valve $V_1$, and during each upstroke the water is carried up until it escapes through the spout $LQ$.

If the height of $DL$ is less than that of the water barometer, the water below the piston will follow it right upto the level $L$. For the effective working of the pump the total height of $DL$ is generally kept a little less than the height of the water barometer.

**7·312. Tension of the piston rod.** Assuming that the piston is moving uniformly and the effects of its weight and friction are not taken into account, the tension of the piston rod will be given by the following:—

*The tension of the piston rod is equal to the weight of a cylinder of water, whose cross-sectional area is equal to that of the piston, and whose height is the same as the vertical distance between the levels of the water within and without the pump.*

Referring to the figure of 7·31, let $K$ be the area of the piston, $h$ the height of the water barometer, and $w$ the weight of a unit volume of water. Let $T$ denote the tension of the piston rod.

The tension of the piston rod must be such that it would balance the difference of pressures on the upper and the lower surfaces of the piston.

Case I. Suppose the water level is at the point $M$ lying between $D$ and $B$.

The pressure of the air above $M$

$$= \text{pressure of the water at level } M$$
$$= \text{pressure at } D - w.DM$$
$$= w.(h - DM).$$

Thus the pressure on the lower surface of the piston

$$= K \times w.(h - DM),$$

and that on the upper surface

$$= K \times wh.$$

Hence, we have

$$T + K \times w(h - DM) = K \times wh,$$

or $$T = K \times w.DM. \quad . \quad . \quad (1)$$

Case II. Suppose the water has risen above the piston to a point $N$.

Now the pressure at a point on the upper surface of the piston

$$= w.EN + wh = w(h + EN).$$

The pressure at a point on the lower surface

$$= wh - w.DE = w(h - DE).$$

Hence

$$T + K \times w(h - DE) = K \times w(h + EN)$$

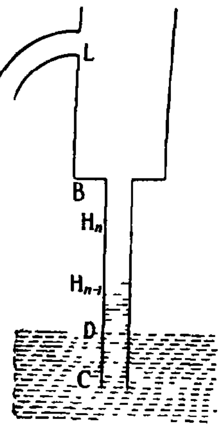
or $$T = K \times w.DN. \quad . \quad . \quad (2)$$

From (1) and (2), the truth of the proposition is demonstrated.

7·313. **Water level raised during the *n*th stroke.** It is required to find the vertical height through which the water is raised by a suction pump during the $n$th stroke of the piston.

Let $DB = a$ and $BL = b$, and suppose the cross-sections of the upper barrel and the suction-tube are $K$ and $k$ respectively. Let $h$ be the height of the water barometer and $w$ the weight of unit volume of water.

Let $H_{n-1}$ denote the water level at the beginning of the $n$th stroke and suppose $H_n$ is the point to which the water is raised at the end of this stroke. If $DH_{n-1} = x_{n-1}$ and $DH_n = x_n$, then the water is raised through the distance $x_n - x_{n-1}$ during the $n$th stroke of the piston.




Case I. Suppose $H_n$ is in the suction-tube.

At the beginning of the $n$th stroke, when the water level is at $H_{n-1}$ and the piston at $B$, the volume of the air in the tube is $(a - x_{n-1})k$ and its pressure $w(b - x_{n-1})$.

At the end of the upward stroke when the position of the piston is at $L$ and the water level at $H_n$, the air occupies the volume $(a - x_n)k + bK$, and its pressure is $w(b - x_n)$.

Hence from Boyle's law, we have after cancelling the common factor $w$

$$[(a - x_n)k + bK](b - x_n) = [(a - x_{n-1})k](b - x_{n-1}). \quad (1)$$

From the above equation $x_n$ can be obtained when $x_{n-1}$ is known. Thus the value of $x_n - x_{n-1}$, that is, the rise in the water-level during the $n$th stroke, can be obtained.

Giving to $n$ in succession the values 1, 2, 3, ... in (1) and noting that $x_0 = 0$, we have

$$[(a - x_1)k + bK](b - x_1) = akb,$$
$$[(a - x_2)k + bK](b - x_2) = (a - x_1)k(b - x_1),$$
$$[(a - x_3)k + bK](b - x_3) = (a - x_2)k(b - x_2),$$

and so on.

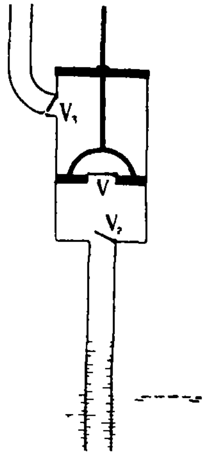
Case II. Suppose $H_n$ is in the upper barrel and $H_{n-1}$ in the suction-tube.

In this case, at the end of the upward stroke the volume occupied by the air is $(a+b-x_n)K$ and its pressure is $w(b-x_n)$. Hence the formula corresponding to (1) will be

$$(a+b-x_n)K.(b-x_n)=(a-x_{n-1})k.(b-x_{n-1}), \quad (2)$$

from which the value of $x_n$ can be obtained in terms of $x_{n-1}$.

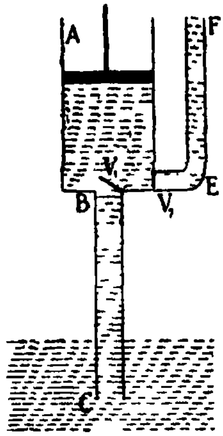
**7·32. The Lifting Pump.** If it be the object to lift water to a high level above the pump, some modification is made in the common pump. The spout is removed, the top of the upper barrel is closed and the piston rod works through a closely fitting tight collar which would allow neither air nor water to pass through. A pipe leading upwards, and generally of smaller section than the spout of the common pump, is fixed to the upper barrel just below its top; at the junction there is a third valve $V_3$ which opens *upwards*. This modified pump is called the *lifting pump*. Theoretically the pump would work in the absence of the valve $V_3$, but in practice it is found that this additional valve helps the action of the valve $V_2$.




After the water has risen above the piston, with each upward stroke the valve $V_3$ opens and water enters the pipe. During the downward stroke o the piston the valve $V_3$ closes and no water can descend to the barrel. In this way water can be lifted to a great height depending upon the strength of the pump and the operator.

**7·33. The Forcing Pump.** This is another modification of the common pump. In this the piston is

solid and has no valve. But there is a second valve $V_2$ at the bottom of the upper barrel opening *outwards* and leading to a pipe $EF$

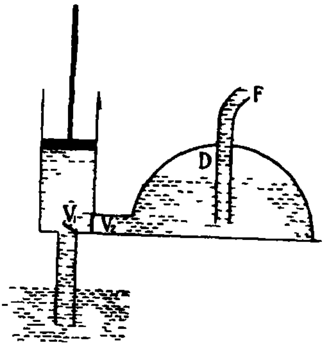



When the piston descends, the valve $V_1$ closes and the air is driven away through the valve $V_2$. During the upward motion of the piston the valve $V_2$ is closed, $V_1$ opens and water rises in $BA$ as in the case of the common pump.

During the next downward stroke of the piston the valve $V_1$ is closed, and all the water contained in the barrel is forced up the pipe $EF$ and is prevented from returning by the valve $V_2$. In this way water can be forced up to any height consistent with the strength of the machine.

The flow of water in a forcing pump of this type will be only intermittent, because water will flow only during the downstrokes of the piston.

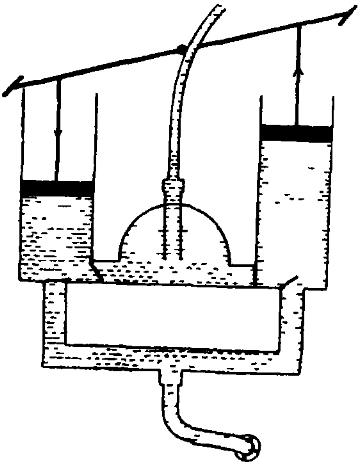
**7·331. The Forcing Pump with Air-chamber.** In order to obtain a continuous stream of water from the top of the pipe $EF$, the water passing through $V_2$ is first admitted into a chamber $D$ containing air.




The piston in its descending stroke forces water into $D$ and the air imprisoned in the chamber is compressed. As more water is forced into the chamber, the air is further compressed, and the water rises in the tube $EF$ and flows out from $F$. When the piston is ascending, the air in $D$,

freed from its former pressure, now expands and forces the water up the tube, thus causing a steady flow of water through the pipe $EF$.

**7·34. The Fire Engine.** This is essentially a double forcing pump connected with the same air chamber. The constancy of the flow is maintained not only by the air chamber, but also by the alternate action of the two pumps. The working of the two pistons is so arranged that while the one ascends, the other descends.



**7·35. Illustrative Example.** *The length of the suction-tube of a common pump is* 12 *ft., and the piston when at its lowest point is* 2 *ft. from the fixed valve. If at the first stroke the water rises* 11 *ft. in the tube, find the extreme length of the stroke, supposing the water barometer to stand at* 33 *ft., and the area of the barrel to be three times that of the tube.*

Let $A$ be the sectional area of the suction-tube, and, therefore, that of the barrel will be $3A$. Then the volume originally occupied by the air is

$$12A + 2.3A = 18A.$$

If $x$ denotes the length of the stroke, the volume occupied by the air after the first stroke is

$$A + (x + 2).3A = (3x + 7)A,$$

and the pressure of this air is reduced to

$$(33 - 11) \text{ ft.} = 22 \text{ ft.}$$

Hence, by Boyle's law, we have

$$33 \times 18A = 22 \times (3x + 7)A,$$

or

$$27 = 3x + 7,$$

or

$$x = 20/3 \text{ ft} = 6 \text{ ft. } 8 \text{ inches.}$$

## Examples XVII

1. In a common pump the length of the barrel is 18 inches, and that of the lower pipe is 21 ft. above the surface of the water; if the section of the pipe is 3/14 of that of the barrel, find the height through which water would rise at the end of the first stroke, taking the height of the water barometer to be 32 feet. [*Patna*, 1931]

2. The suction-tube of a common pump is 12ft. long, and the piston starting from the fixed valve is raised at the first stroke through 3 ft. If the area of the barrel is four times that of the tube, find the height to which water will be raised in the tube, water barometer being 34 ft. high.

3. The length of the lower pipe of a common pump above the surface of the water is 10 ft., and the area of the section of the upper pipe is 4 times that of the lower. Taking 33 ft. as the height of the water barometer, prove that if at the end of the first stroke the water just rises into the upper pipe, the length of the stroke must be very nearly 3 ft. 7 inches. [*Allahabad*, 1921]

4. If the length of the lower pipe of a common pump above the surface othe water be 16 ft., and the area of the barrel of the pump 16 times that of the pipe, find the length of the stroke so that the water may just rise into the barrel at the end of the first stroke, the water barometer standing at 32 ft. If the length of the stroke of the piston be one foot, find the height to which the water will rise at the end of the first stroke.

5. The length of the lower pipe of a common pump above the surface of water is 20 ft., the cross-section of the barrel of the pump is 36 times that of the pipe, and the length of the stroke is 1 ft. Find how far the water will rise at the end of the first stroke, taking the height of the water barometer to be 32 ft. [*Patna*, 1934]

6. A lifting pump is employed to raise water through a vertical height of 200 ft. If the area of the piston be 100 sq. inches, what is the greatest force, in addition to its own weight that will be required to lift the piston?

7. If $A$ be the area of the section of the piston of a force pump, $l$ the length of the stroke, $n$ the number of strokes per minute, $B$ the area of the pipe from the pump, find the mean velocity with which the water rushes out. [*M. T.*]

8. A forcing pump the diameter of whose piston is 6 inches, is employed to raise water from a well to a tank. If the bottom of the

piston is 20 ft. above the surface of the water in the well and 100 ft. below the surface in the tank, find the least force which will (i) raise, (ii) depress the piston; friction and the weights of the valves being neglected, and the height of barometer taken as 32 ft.

9. Describe any arrangement by which water could be pumped up from a well 50 ft. deep, and explain clearly the mode of action of such an instrument.

The lower valve of a common pump is a piece of brass 8 ozs. in weight, resting over a hole 2 sq. inches in area. Find the greatest height at which the pump can be installed, if the height of the water barometer is 34 ft. and the weight of one cubic foot of water is 1000 ozs. [*Lucknow*, 1939]

10. Prove that if $h$, $h'$ be the heights at which the water stands in the lower cylinder of a common pump before and after a stroke, then

$$(h'-h)(h'+h-H-a)+nb(H-h')=0,$$

where $a$, $b$ are the lengths of the lower and upper cylinders, $n$ is the ratio of the sectional area of the latter to that of the former, and $H$ is the height of the water barometer.

11. If the barrel of a common pump be 2 ft. long and its lower end be 26 ft. above the surface, and if the area of the section of the barrel be 6 times that of the pump, find in how many strokes the water will reach the barrel, the height of the water barometer being 32 ft.

12. Prove that in the common pump the water will just rise into the upper cylinder at the end of the second stroke, if

$$H^2(1-a/nb)(2-a/nb)-H(4a+nb-3a^2/nb)+a(2a+nb)=0,$$

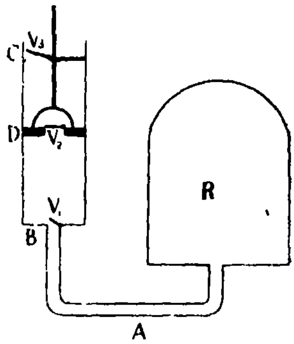
where $a$, $b$ are the lengths of the lower and upper cylinders, $n$ is the ratio of the sectional area of the latter to that of the former and $H$ is the height of the water barometer. [*M. T.*]

## Air-Pumps

**7·4. Exhausting and Condensing Pumps.** The use of air-pumps is either to remove air from a vessel or to force into it some air from outside. Those belonging to the latter type are generally called *Compressing* or *Condensing pumps* or *Condensers*, whilst the name air-pump is usually restricted to those of the former class. Great care has to

be taken in air-pumps so that the fittings and valves are all air-tight.

**7·41. Smeaton's Air-Pump.** This pump consists of a vessel R, called the *receiver*, from which the air is to be exhausted; of a cylinder or barrel *BC*, furnished with a piston and valves; and of a pipe *A* which serves to connect the receiver with the cylinder. There are three valves all opening *upwards*, $V_1$ at the bottom and $V_3$ at the top of the cylinder, while the third valve $V_2$ is fitted in the piston *D* itself.



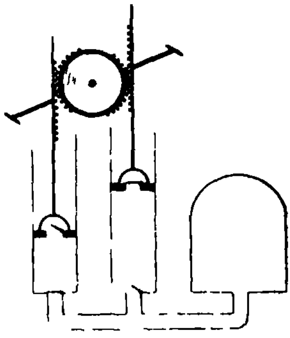

To explain the action of the pump suppose the working is commenced when the piston is at its lowest position, *B*. When the piston is raised, a partial vacuum is formed between it and *B*, the valve $V_1$ opens and air from the receiver enters into the cylinder. The air which is above the piston becomes condensed, causes the valve $V_3$ to open and escapes out into the atmosphere. Thus a portion of air occupying the volume of the cylinder at atmospheric pressure is removed in the first upstroke of the piston.

When the piston has reached *C*, its motion is reversed. The air lying between the piston and *B* becomes compressed and causes the valve $V_1$ to close and $V_2$ to open. The air from below the piston passes up through the valve $V_2$ and occupies the space above the piston. When the piston returns to *B*, the air of the cylinder which was below it at the end of the first upstroke, comes above the piston. During the next upward stroke this volume of air passes out into the atmosphere and the cylinder is again filled up with air from the receiver.

In this way in each succeeding stroke of the piston a volume of air equal to that of the cylinder, but at diminishing pressure, is removed from the receiver. The process can be continued until the pressure of the air left in the receiver is reduced to such an extent that it is unable to lift the valve $V_1$.

In practice it is found that the piston does not generally descend to the extreme bottom of the barrel so that an upstroke invariably begins with some air lying between the piston and the lowest valve. The volume of the barrel from the bottom to the point where the piston can reach in its lowest position, is called a '*clearance*.'

**7·42. Hawksbee's Air-Pump.** This pump is essentially the same as the Smeaton's pump, except that in this case the top of the cylinder is kept open. The pump is usually constructed with two barrels furnished with a piston in each. The pistons are worked by means of a toothed wheel in such a manner that the upstrokes of the one synchronize with the downstrokes of the other.



The Smeaton's pump on account of having a valve at $C$ [Fig. of 7·41] has certain advantage over the Hawksbee's pump. During the downstroke of the piston, the pressure of the air above it being less than that of the atmosphere, the piston valve in the Smeaton's pump is raised more easily than in the case of the Hawksbee's pump. Also the effort required to lift the piston is decreased to a great extent. The exhaustion of the air can be carried out more effectively by the Smeaton's pump.

**7·43. Rate of Exhaustion of the air.** In the figure of 7·41, suppose $V$ is the volume of the receiver including that of the pipe, and $V'$ is the volume of the cylinder.

Let $\rho$ be the initial density of the air in the receiver and $\rho_1$ the density after the first upward stroke. Since the air which originally occupied a volume $V$ and had density $\rho$ now occupies a volume $V + V'$ and has density $\rho_1$, we have

$$V \cdot \rho = (V + V') \cdot \rho_1.$$

$$\therefore \qquad \rho_1 = \frac{V}{V + V'} \cdot \rho \cdot \qquad . \quad . \quad . \quad (1)$$

At the end of the first complete stroke, a volume $V'$ of air of density $\rho_1$ has escaped and there remains in the receiver a volume $V$ of air of density $\rho_1$.

If $\rho_2$ be the density of the air of the receiver after the second complete stroke, then as explained in the case of the first complete stroke, we get

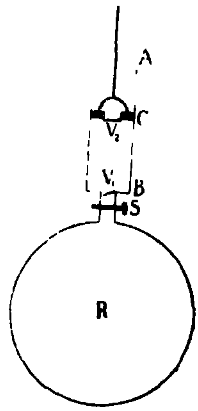
$$\rho_2 = \frac{V}{V + V'} \cdot \rho_1 = \left(\frac{V}{V + V'}\right)^2 \cdot \rho \cdot \qquad . \quad . \quad . \quad (2)$$

Repeating the same process, if $\rho_n$ be the density of the air of the receiver after the $n$th stroke, then

$$\rho_n = \left(\frac{V}{V + V'}\right)^n \cdot \rho \cdot \qquad . \quad . \quad . \quad . \quad (3)$$

It may be noted that since $\rho_n$ can never be zero, a complete vacuum cannot be obtained in this manner.

**7·44. The Air-Condenser.** If the direction of motion of the valves of an air-pump be reversed, air will be forced into the receiver by the motion of the piston and the pump will become a *Condenser*.




The condenser consists of a cylinder or barrel $AB$, into which works a piston $C$ having in it a valve $V_2$ that opens *downwards*. At the bottom of the cylinder there is another valve $V_1$ that likewise opens *downwards* and communicates with the receiver $R$ by means of a tube which is fitted with a stop-cock $S$.

When the piston moves down, the valve $V_2$ is closed, the air between the piston and $B$ is compressed, the valve $V_1$ opens and the air is driven into the receiver. When the piston has reached $B$, its motion is reversed A partial vacuum is created between $B$ and the piston, the valve $V_2$ opens, but the increased pressure of the air in the receiver closes the valve $V_1$ and the air of the receiver is prevented from re-entering the barrel

Thus in one complete stroke a volume of atmospheric air equal to that of the cylinder has been forced into the receiver In succeeding strokes more and more air from the atmosphere is forced into the receiver

7·45. **Density of the air in the Condenser.** In the figure of 7·44, suppose $V$ is the volume of the receiver and $V'$ that of the cylinder

Since in each stroke of the piston a volume $V'$ of air at atmospheric pressure is pumped into the receiver, at the end of the $n$th stroke the receiver will contain a quantity of air which would occupy a volume $V + nV'$ at atmospheric pressure Hence, if $\varrho$ be the initial density of the air of the receiver and $\varrho_n$ its density after the $n$ strokes of the piston, we have

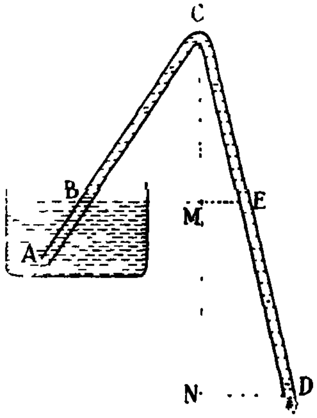
$$\varrho\,(V + nV') = \varrho_n\, V,$$

or

$$\varrho_n = \frac{V + nV'}{V} \cdot \varrho. \qquad (1)$$

7·5. **The Siphon** The siphon is an appliance for extracting liquid from a vessel without moving the vessel. It consists simply of a bent tube, open at both ends, and with arms of unequal length. In order to work the siphon it is first filled with the same kind of liquid as is to be extracted, and then, closing the two ends $A$ and $D$ the siphon, is inverted and placed with the end $A$ of

the shorter arm below the surface of the liquid in the vessel, as is shown in the adjoined figure. The end $D$ of the longer arm is kept below the level of the liquid in the vessel. Now when the ends are opened, the liquid begins to flow out in a continuous stream and will continue to do so till the level of the liquid in the vessel falls below $A$, provided the vertical distance between $A$ and the top $C$ be less than the barometric height for that liquid.




To explain the action of the siphon, let $CM$ and $CN$ represent the vertical heights of the top $C$ above the surface level $B$ in the vessel and above $D$ respectively. The pressure at $B$ being the atmospheric pressure $\Pi$, the pressure at $C$ is $\Pi - g\rho.CM$, where $\rho$ denotes the density of the liquid. Therefore the pressure in the liquid at $D$ is

$$\Pi - g\rho.CM + g\rho.CN$$
$$= \Pi + g\rho \cdot MN,$$

which exceeds the atmospheric pressure, and consequently the liquid at $D$ flows out followed by the liquid from the vessel through the tube.

**7·6. Illustrative Examples.** (i) *The capacity of the receiver of a Smeaton's air-pump is 9 times that of the barrel; determine* (a) *the density of the air in the receiver after 5 strokes,* (b) *the number of strokes required to reduce the density of the air to 0·1 of its original value.*

Let the volume of the barrel be $V$, then that of the receiver is $9V$. Thus a mass of air which occupies a volume $9V$ before a stroke, occupies a volume $10V$ at the end of that stroke. Hence after each stroke the density is reduced in the ratio $9:10$.

(a) Thus after 5 strokes the density of the air

$$= (9/10)^5, \text{ i.e., } 0{\cdot}59 \text{ approx., of its initial density.}$$

(b) Suppose the density of the air is reduced to $0 \cdot 1$ of its original value after $n$ strokes. Then from (3) of 7·43, we get

$$1/10 = (9/10)^n.$$

Taking logarithms, we have

$$-1 = n(\log 9 - 1)$$
$$= n(0 \cdot 9542 - 1)$$
$$= -n \times 0 \cdot 0458.$$

$$\therefore \qquad n = \frac{1}{0 \cdot 0458} = 21 \cdot 83,$$

which means that the required value of the density will be reached during the twenty-second stroke.

(ii) *The barrel of a condensing air-pump is one inch in diameter and* 16 *inches long. The tube of a pneumatic tyre in its inflated conditions is* 2 *inches in diameter and* 80 *inches long. Assuming the tube to be quite empty initially, how many strokes will be required to inflate it with air at twice the atmospheric pressure?*

If $V$ be the volume of the tube and $V'$ that of the barrel of the condenser, then

$$V = \pi . 1 . 80 \,; \; V' = \pi \cdot (1/2^2) \times 16. \qquad \ldots \quad (1)$$

Now if $n$ be the required number of strokes and $\Pi$ the atmospheric pressure, we have from (1) of 7·45

$$2\,\Pi = \frac{V + nV'}{V} . \Pi,$$

or $$1 + n \cdot V'/V = 2,$$

or $$n \cdot 1/20 = 1, \quad \text{from (1)}$$

or $$n = 20.$$

## Examples XVIII

1. Describe Hawksbee's air-pump and find the density and pressure of the air left in the receiver after $n$ strokes.

The volumes of the barrel and the receiver are 25 and 75 c. inches respectively. Find the pressure of the air after 3 strokes.

[*Bombay*, 1935]

2. If the volume of the receiver in a Smeaton's air-pump is 5 times that of the barrel, find the pressure in the receiver after 3 strokes of the piston, the barometric height being 30 inches. [*Calcutta*, 1912]

3. In one pump the barrel has $1/12$ th of the volume of the receiver and in another it has $1/6$ th. How many strokes of the latter

are required to produce the same degree of exhaustion as six of the former? [*Patna*, 1935]

4. If the receiver of an air-pump be 6 times as large as the barrel, find the number of strokes required before the density of the air is less than 1/4th of the original density. [*Patna*, 1941]

5. The capacity of the receiver of a Smeaton's air-pump is eight times that of the barrel; what fraction of the 5th upward stroke has the piston described when the upper valve opens?

6. A condenser and a Smeaton's air-pump have equal barrels and a receiver whose volume is 20 times that of each barrel, is connected to the condenser. If after the condenser has been worked for 20 strokes, the receiver is connected with the air-pump, show that after 14 strokes the density of the air in the receiver will be nearly the same as it was before the condenser was worked. (Given that $\log 2 = 0 \cdot 3010$, $\log 21 = 1 \cdot 3222$.)

7. The mass of air in the receiver of an air-pump is $m$ at the beginning, and it becomes $m'$ after $n$ complete strokes of the piston. If $V$ and $v$ denote the volumes of the receiver and the barrel, prove that

$$n = \frac{\log m - \log m'}{\log(V + v) - \log V}.$$ [*Allahabad*, 1938]

8. The barrel of an air-pump is of volume $B$, but of this a volume $b$ is not traversed by the piston; show that the density in the receiver never becomes as small as $b/B$ of the atmospheric density.

9. If of the volume $B$ of the cylinder of a condenser only $C$ is traversed by the piston, prove that the pressure in the receiver cannot be made to exceed $B/(B-C)$ atmospheres.

10. If $h$ be the range of the piston in a Smeaton's air-pump, $a$ its distance from the top of the barrel in its highest position, $b$ its distance from the bottom in its lowest position, and $\varrho$ the density of the atmosphere, prove that the limiting density of the air in the receiver will be

$$\frac{ab}{(h + a)(h + b)}\,\varrho.$$ [*M. T.*]

11. Describe an air-pump and explain its action.

A body weighing $n$ lbs. in air weighs $(n + 1)$ lbs. in the receiver of an air-pump after $n$ strokes of the piston. If the capacity of the receiver is $n$ times that of the barrel, find the weight of the body in vacuum. [*Lucknow*, 1939]

# MISCELLANEOUS EXAMPLES

1. A vessel $A$ contains a quantity of liquid of sp gr $\varrho$, and a second cask an equal quantity of liquid of sp gr $\sigma$, one $n$th part of each is taken out and put into the other and well mixed, the process is repeated $m$ times, show that the final sp. gravities are

$$\varrho + \frac{\sigma - \varrho}{2}\left\{1 - \left(1 - \frac{2}{n}\right)^m\right\} \text{ and } \sigma + \frac{\varrho - \sigma}{2}\left\{1 - \left(1 - \frac{2}{n}\right)^m\right\}.$$

2 The two arms of a U-tube are close together In one arm there is water and in the other mercury, so that their common surface is at the lowest point One quarter of the water is taken out and is poured into the other arm over the mercury Prove that in the new equilibrium position the difference of heights of the upper surfaces is one half of what it was formerly

3. A fine tube bent in the form of an ellipse is held with its plane vertical and is filled with $n$ liquids whose densities are $\varrho_1, \varrho_2, \ldots, \varrho_n$ taken in order round the elliptic tube If $r_1, r_2, \ldots, r_n$ be the distances of the points of separation from either focus, prove that

$$r_1(\varrho_1 - \varrho_2) + r_2(\varrho_2 - \varrho_3) + \ldots + r_n(\varrho_n - \varrho_1) = 0$$

4. A circular tube, centre $O$, is filled with three liquids of densities $\varrho_1, \varrho_2, \varrho_3$ in descending order of magnitude and placed in a vertical plane. If $2\alpha$, $2\beta$, $2\gamma$ be the angles subtended at the centre by the fluids and $P$ the point on the circumference, midway between the ends of the lightest fluid, prove that the angle $\theta$ which $OP$ makes with the vertical is given by

$$\frac{\varrho_2 - \varrho_3}{\varrho_1 - \varrho_3} = \frac{\sin\alpha}{\sin(\alpha+\theta)} \cdot \frac{\sin(\beta-\theta)}{\sin\beta}.$$

5. If three liquids which do not mix and whose densities are $\varrho_1, \varrho_2, \varrho_3$, fill a circular tube in a vertical plane and if $\alpha, \beta, \gamma$ are the angles which the radii to the common surface make with the vertical diameter measured in the same direction, prove that

$$\varrho_1(\cos\beta - \cos\gamma) + \varrho_2(\cos\gamma - \cos\alpha) + \varrho_3(\cos\alpha - \cos\beta) = 0.$$

If there are equal volumes of each fluid, and if in addition the weights on each side of the vertical are equal, obtain an equation to determine $\alpha$, corresponding to the highest junction. Show that it is satisfied if $\alpha = 30°$, and the densities are in A. P.

6. A V-shaped trough consists of two rectangular sides equally inclined to the horizontal and two vertical triangular ends. A given volume of liquid is poured into the trough. Determine the angle between the sides which makes the ratio of the thrust on each end to that on a side a maximum. [*M. T.*]

7. A parallelogram is immersed in a fluid with a diagonal vertical, one extremity of which is in the surface of the fluid. Through this point lines are drawn dividing the parallelogram into three equal parts. Compare the pressures on these three parts, and if $P_2$ be the pressure on the middle part and $P_1$, $P_3$ those on the other two, prove that $16P_2 = 11(P_1 + P_3)$.

8. A vessel contains $n$ different liquids resting in horizontal layers and of densities $\rho_1, \rho_2, \ldots\ldots, \rho_n$, starting from the highest fluid. A triangle is held with its base in the upper surface of the highest fluid, and with its vertex in the $n$th fluid. Prove that if $\Delta$ be the area of the triangle and $h_1, h_2, \ldots, h_n$ be the depths of the vertex below the upper surfaces of the 1st, 2nd, ..., $n$th fluids respectively, the thrust on the triangle is

$$\frac{1}{3}\frac{g\Delta}{h_1^2}\{\rho_1(h_1^3 - h_2^3) + \rho_2(h_2^3 - h_3^3) + \ldots\ldots + \rho_n h^3_n\}.$$

9. Find the force on the vertical end of a parabolic gutter if it is full of water, the gutter being 6 inches across the top and 4 inches deep.

10. A solid triangular prism, the faces of which include angles $\alpha$, $\beta$, $\gamma$ is placed in any position entirely within a fluid; if $P$, $Q$, R be the thrusts on the three faces respectively opposite to the angles $\alpha$, $\beta$, $\gamma$, prove that

$$P \operatorname{cosec} \alpha + Q \operatorname{cosec} \beta + R \operatorname{cosec} \gamma$$

is invariable so long as the depth of the centre of gravity of the prism is unchanged. [*M. T.*]

11. A tunnel of rectangular section, of height $h$ feet, is closed by a heavy uniform metal door, inclined at an angle $\alpha$ to the vertical, and swung on hinges along the roof of the tunnel. Show that if the door is to open automatically just when the level of water in the tunnel rises to the roof, the weight per square foot of the door must be equal to that of $\frac{2}{3}h \operatorname{cosec}\alpha$ cubic feet of water. [*M.T.*]

12. One wall of a tank slopes inwards from the bottom at an angle $\theta$ to the vertical and contains a triangular trap-door of weight $W$, which is hinged about a horizontal side $BC$, has the vertex $A$ lower

than $BC$, and can open outwards. The vertical heights of the vertices above the bottom of the tank are $a$, $b$, $b$. Prove that, if water be poured into the tank to a height $h$ so that the trap-door is entirely below the surface, it will remain closed, provided that

$$h < \frac{W}{s\Delta} \sin \theta + \tfrac{1}{2}(a + b),$$

$\Delta$ being the area of the triangle and $s$ the weight of the unit volume of water. [*M.T.*]

13. A portion of a sphere cut off by two planes through its centre, inclined to each other at an angle $\pi/4$, is just immersed in a liquid with one face in the surface. Find the resultant thrust on the curved surface and show that it makes an angle $\tan^{-1}(\pi/2 - 1)$ with the horizontal. [*I. C. S.*, 1937]

14. A solid is formed by the revolution of a semi-circle of radius $a$ about its bounding diameter through an angle $\alpha$, and the solid is immersed with one plane face in the surface of a liquid; prove that the magnitude of the resultant thrust on the curved surface of the solid is

$$\tfrac{2}{3}a^3 g\rho \{(\alpha - \sin \alpha \cos \alpha)^2 + \sin^4 \alpha\}^{1/2},$$

where $\rho$ is the density of the liquid.

15. A right circular cone is held in a liquid with its axis horizontal, and the highest point $C$ of its base in the surface Find the magnitude and direction of the resultant pressure on the curved surface, and determine the angle of the cone when the line of action of this pressure (i) passes through $C$, (ii) is parallel to a generating line.

16. A right cone of semi-vertical angle $\alpha$ is just immersed with a slant side of length $l$ in the free surface Show that the resultant thrust on the curved surface will cut this slant side at a distance

$$\frac{3b}{4(1 - 3 \sin^2 \alpha)}$$

from the vertex, and find the magnitude of this thrust.

[*Bombay*, 1940]

17. Given that the centre of pressure of a circular disc of radius $r$ with one point in the surface, is at a distance $p$ from the centre, show that for a disc of radius R wholly immersed with its centre at a distance $h$ from the surface, the distance between the centre of the circle and the centre of pressure is $pR^2/hr$. [*M. T.*]

18. Find the centre of pressure of a triangle immersed with one angular point in the surface and the other two angles at depths

$\alpha$ and $\beta$ below the surface of a liquid whose density varies as the depth. [*Patna*, 1931]

19. A semi-circular lamina is completely immersed in water with its plane vertical, so that the extremity $A$ is in the surface, and the diameter makes with the surface an angle $\alpha$. Prove that if $E$ be the centre of pressure, and $\theta$ the angle between $AE$ and the diameter,

$$\tan\theta = \frac{3\pi + 16\tan\alpha}{16 + 15\pi\tan\alpha}.$$ [*I. C. S.*, 1936]

20. A circular disc of radius $a$ is completely immersed with its plane vertical in a homogeneous fluid. If $b$ is the depth of the centre below the free surface of the fluid, prove that the distance between the centres of pressure of the two semi-circles into which the disc is divided by its horizontal diameter is

$$6\pi a\,\frac{4b^2 - a^2}{9\pi^2 b^2 - 16a^2}.$$ [*M. T.*]

21. If a plane regular pentagon is immersed so that one side is horizontal and the opposite vertex at double the depth of that side, prove that the depth of the centre of pressure of the pentagon is $a(29 + 3\sqrt{5})/48$, where $a$ is the depth of the lowest vertex

22. A triangular lamina $ABC$ right-angled at $C$, is just immersed in a fluid with the vertex $C$ in the surface and the side $CA$ inclined at $30°$ with the surface. If the centre of pressure is vertically below $C$, prove that the angle $B$ is $\frac{1}{2}\tan^{-1}(2\sqrt{3})$. [*Nagpur*, 1940]

23. A semi-ellipse bounded by its minor axis, is just immersed in a liquid the density of which varies as the depth. If the axis minor be in the surface, find the eccentricity in order that the focus may be the centre of pressure.

24. An area bounded by the curve $ay^2 = x^3$, the $x$-axis and the ordinate $x = b$, is immersed in water with the ordinate in the surface. Find the coordinates of the centre of pressure.

25. A quadrant of a circle is just immersed vertically with one edge on the surface in a liquid whose density varies as the depth. Obtain the centre of pressure.

26. A steamer loading 30 tons to the inch near the water line in fresh water is found after a ten days' voyage, burning 60 tons of coal a day, to have risen 2 feet in sea-water at the end of the voyage; prove that the original displacement of the steamer was 5720 tons, taking a cubic foot of fresh water as 62.5 lbs. and of sea-water as 64 lbs.

27. A sphere of radius $a$ and mass $M$ is loaded so that its centre of gravity $G$ is at a distance $C$ from its centre $O$, and is suspended by a string attached to a point $P$ of its surface, $GP$ subtending an angle $\theta$ at $O$. The sphere is partially immersed in a liquid of density $\rho$ and the tension in the string is $M'g$. Show that the depth $b$ of $O$ below the surface of the liquid is given by

$$M - M' = (\pi\rho/3)(a+b)^2(2a-b),$$

and that the inclination of $GO$ to the vertical is

$$\tan^{-1}\frac{M'a\sin\theta}{MC - M'a\cos\theta}. \qquad [M.\ T.]$$

28. $ABC$, an isosceles triangle right-angled at $A$, composed of two heavy rods $AB$, $AC$ hinged together at $A$ and a light string $BC$, floats with the angle $A$ immersed in water. Show that the tension of the string is $(a-b)W/2a$, where $2a$ is the length of a rod, $2b$ the length immersed and $W$ the weight of each rod.

[*Bombay*, 1940]

29. An equilateral triangle $ABC$, of weight $W$ and sp. gr. $\sigma$, is movable about a hinge at $A$ and is in equilibrium when the angle $C$ is immersed in water and the side $AB$ is horizontal and above water. It is then turned about $A$ in its own vertical plane until the whole of the side $BC$ is in the water and horizontal, prove that the pressure on the hinge in this position is

$$2\left(\frac{1-\sqrt{\sigma}}{\sqrt{\sigma}}\right)W. \qquad [M.\ T.]$$

30. Two equal and similar rods $AB$, $BC$, fixed at an angle $\alpha$ at $B$, rest in a fluid of twice the sp. gr. with the angle $B$ out of the fluid, and the bisector of the angle $ABC$ makes an angle $\theta$ with the horizon, prove that $\cos 2\theta + \sec\alpha = 2$. [*Bombay*, 1937]

31. A cone is suspended by its vertex from a point above the surface of a liquid and rests with a generating line vertical. If the vertical angle of the cone be 60°, and the height of the vertex above the liquid equal to the radius of the base, prove that the densities of the cone and liquid are in the ratio of $2\sqrt{2}-1 : 2\sqrt{2}$. [*M. T.*]

[The section by the water surface is an ellipse whose area is $\pi c^2\sin^2\alpha\cos\alpha/(\cos 2\alpha)^{3/2}$, where $c$ is the distance of the elliptic section from the vertex of the cone and $2\alpha$ the vertical angle of the cone.]

32. A semi-circular lamina has one of the ends of its diameter smoothly hinged to a fixed point above the surface of a liquid and

floats with its plane vertical and its diameter half-immersed. If the inclination of the diameter to the horizon is $\pi/4$, prove that the ratio of the density of the liquid to that of the lamina is $4(3\pi - 4) : (9\pi - 8)$. [*M. T.*]

33. *ABC* is a right-angled triangular lamina and it floats with its plane vertical, and the right angle immersed in water; prove that if its sp. gr. be to that of the water as $2 : 5$, and $CB : CA = 5 : 4$, $CB$ is cut by the surface of the water at a distance from $C$ equal to $CA$.

34. A right cylinder of radius $a$ and height $2b$ floats in a fluid of double its density with one of its circular ends entirely out of the fluid; show that it can rest with its axis inclined at a certain angle to the vertical if $b > a/\sqrt{2}$.

35. A square lamina is placed vertically in a fluid of double its density; prove that it can rest only with one edge or diagonal vertical. [*M. T.*]

36. A wooden sphere of radius $r$ is held just immersed in a cylindrical vessel of radius R containing water, and is allowed to rise gently completely out of the water; prove that the loss of the potential energy of the water is

$$\frac{Wr(3R^2 - 2r^2)}{3R^2},$$

$W$ being the weight of water displaced by the sphere.

37. A sphere of radius $a$ and sp. gr. $\frac{1}{2}$, is held completely immersed at the bottom of a circular cylinder of radius $b$ which is filled with water to depth $d$. The sphere is set free and takes up its position of equilibrium. Show that the potential energy lost is

$$W\left(d - \frac{11}{8}a - \frac{a^3}{3b^2}\right),$$

where $W$ is the weight of the sphere. [*I. C. S.*, 1943]

[Potential energy is gained by the sphere in rising from the bottom of the cylinder to the surface of the water, but it is lost by the water whose centre of gravity is lowered.]

38. A cylindrical piece of wood of length $l$ and sectional area $\alpha$ is floating with its axis vertical in a cylindrical vessel of sectional area $A$ which contains water; prove that the work done in very slowly pressing down the wood until it is just completely immersed, is

$$\tfrac{1}{2}g\alpha l^2\left(1 - \frac{\alpha}{A}\right)\frac{(\varrho - \sigma)^2}{\varrho},$$

where $\varrho$ and $\sigma$ denote the densities of the water and wood respectively. [*M. T.*]

39. A lump of matter of mass $m$ floats in a liquid contained in a circular cylinder of radius $r$. The lump melts and forms a layer of liquid on the top of the other. Show that the rise of level of liquid is

$$\frac{m}{\pi r^2}\left\{\frac{1}{\sigma_2} - \frac{\sigma_1 - \varrho}{\sigma_1(\sigma - \varrho)}\right\},$$

where $\varrho$ is the density of the atmosphere, $\sigma_1$, $\sigma_2$ the densities of $m$ when solid and liquid respectively, and $\sigma$ the density of the first liquid. [*Bombay*, 1936]

40. Prove that if volumes $V_1$ and $V_2$ of atmospheric air are forced into vessels of volumes $U_1$ and $U_2$ and if communication is established between them, a quantity of air of volume

$$\frac{U_1V_2 - U_2V_1}{U_1 + U_2}$$

at atmospheric pressure will pass from one to the other.

41. A gas saturated with vapour is at a pressure $\Pi$. It is then compressed without change of temperature to $1/n$th of its former volume, and the pressure is then observed to be $\Pi_n$. Show that the pressure of the vapour is

$$\frac{n\Pi - \Pi_n}{n - 1}$$

and that the pressure of the air in the original volume without its vapour is

$$\frac{\Pi_n - \Pi}{n - 1}.$$

42. A piston of weight $W$ serves as a stopper to confine some gas within a vertical cylinder open at the top. If $h$ is the equilibrium height of the base above that of the cylinder and if it is slowly displaced through a small distance $x$, without any change in the temperature of the gas taking place, show that the work done on the system is $Wx^2/2h$ nearly. [*M. T.*]

43. The tube of a barometer rises to 34 inches above the mercury in the trough, and the mercury column is 30 inches high. Find what changes are produced in the height of the column by the following operations performed successively:—

(1) As much air is allowed to rise through the mercury as would, at atmospheric pressure, occupy 2 inches of the tube.

(ii) A rod of iron whose volume equals that of 5 inches of the tube, is allowed to float at the top of the mercury column.

(Sp. gr. of mercury 13·5 and that of iron 7·5.) [*I. C. S.*, 1937]

44. A piston of weight $w$ rests in a vertical cylinder of transverse section $k$, being supported by a depth $a$ of air. The piston rod receives a vertical blow $P$, which forces the piston down through a distance $b$; prove that $(w + \Pi k)\{b + a \log (1 - b/a)\} + gP^2/2w = 0$, $\Pi$ being the atmospheric pressure.

45. On the assumption that the temperature of the atmosphere is constant and equal to 0°*C.*, prove that if the barometer readings at two stations are $h$ and $h'$ inches, the height of the second station above the first is $[5\sigma/(2s \log_{10} e)] \log_{10} (h/h')$ feet, where $\sigma$ is the sp. gr. of mercury at 0°*C.*, and $s$ that of air at 0°*C.*, when the barometer stands at 30 inches.

46. A thin heavy cylinder, hollow and open at the lower end is found when depressed from the atmosphere successively into three liquids, to remain at rest when its higher end is at respective depths $h$, $h'$, $h''$ below the surfaces. If $s$, $s'$, $s''$ be their specific gravities, prove that, the weight of the air contained in the cylinder being neglected in comparison with that of the cylinder,

$$s(s' - s'')h + s'(s'' - s)h' + s''(s - s')h'' = 0.$$

[*M. T.*]

47. A cylindrical diving bell fully immersed, is in equilibrium without a chain. Show that if the exterior atmospheric pressure increases slightly, the ratio of the distance moved through by the bell, to that moved through by the surface of water in the bell when held fixed, is $Hb + x^2 : x^2$ app., where $H$ is the height of the water barometer, $b$ the height of the bell, and $x$ the height of that part of it which is filled with air.

48. A cylindrical diving bell of height $b$, is immersed in water with its highest point at a depth $a$ below the surface; if the barometer rises so that the increase of the pressure on its top is $P$, show that the alteration in the tension of the chain is approximately

$$\frac{P}{2}\left[1 - \frac{a + b}{\sqrt{(a + b)^2 + 4bh}}\right],$$

where $h$ is the height of the water barometer.

49. If a diving bell in the shape of an inverted cone, of height $a$, be lowered till its vertex is at a depth $d$, prove that the height $x$

of the part of the bell occupied by the air is given by the equation $x^4 + x^3(b + d) = a^3b$, where $b$ is the height of the water barometer.

If the temperature of the air inside be now raised from $T°$ to $(T + t°)$, prove that the tension of the supporting chain is diminished by

$$\frac{3\alpha t b W}{3b + 3a + 4x},$$

where $W$ is the weight of the water the cone would contain, and $\alpha$ the coefficient of expansion, the squares of $\alpha$ being neglected.

50. A body floats in water, the volume of the part not immersed being $cA$. A cylindrical diving bell of height $b$ and cross-section $A$ is placed over it and then lowered till the top of the bell is at a distance $a$ below the surface of the water. The volume of the floating body which is now not immersed is $(c + \gamma\sigma)A$, show that $\gamma$ is the positive root of the equation

$$b\gamma^2 + c(b - a - c)\gamma - c^2(a + b) = 0,$$

where $b$ is the height of the water barometer and $\sigma$ is the sp. gr. of the air, $\sigma$ being small. [*M. T.*]

51. Prove that if the piston of Hawksbee's air pump cannot traverse the whole length of the cylinder, the density in the receiver after $n$ strokes will be

$$1 - \left\{1 - \left(\frac{A}{A + B}\right)^n\right\}\frac{C}{B}$$

of the density of the atmosphere, supposing $A$ to denote the volume of the receiver, $B$ that of the cylinder, and $C$ that of the part traversed by the piston. [*M. T.*; *Patna*, 1941]

52. Show that if the piston valve of a condenser does not open until the pressure on the outside exceeds that on the inside by $k$ times the atmospheric pressure, the number of strokes which can be made before the valve ceases to act is the greatest integer in

$$1 + \left\{\log \frac{A}{k(A + B)}\right\} \Big/ \log \frac{C + B}{C},$$

where $C$ is the volume of the receiver, $A$ that of the portion of the cylinder traversed by the piston and $B$ that of the portion not traversed by it.

APPENDIX A

# DETERMINATION OF SPECIFIC GRAVITY

**A·1. General Considerations.** In 1·4 the *specific gravity* of a substance was defined to be the ratio of the weight of any volume of that substance to the weight of an equal volume of some standard substance, usually water. In order to determine the specific gravity of a given substance we have, therefore, to find the weights of equal volumes of the substance and water. Various methods have been devised for this purpose; in quite a number of them the Principle of Archimedes is made use of. For, if $W$ represents the *real* weight of a body, that is, its weight in vacuum, and $W'$ its *apparent* weight in water, $(W - W')$ is the weight of the water equal in volume to that of the body by Archimedes' Principle, and hence the sp. gr. of the substance will be given by $W/(W - W')$.

The various methods employed for the determination of specific gravities of substances are described in detail in books on *Practical Physics* to which reference may be made, if required. Here we shall briefly explain the principles and methods used in finding the specific gravity by means of the following appliances :—

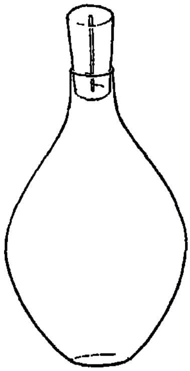
(1) The Specific Gravity Bottle,
(2) The Hydrostatic Balance, and
(3) Hydrometers.

Instead of the weight of the body in vacuum, we generally use its weight taken in air, the difference being immaterial for bodies of small magnitude. But in order to obtain a more accurate result, correction for atmospheric effect must be applied. When a solid is weighed in water, there may arise some discrepancy on

account of the weight of the wire or thread by which it is supported, the capillary force and the dissolved air contained in the water. For extreme accuracy the effect of all these should be taken into account.

**A·2. The Specific Gravity Bottle.** This is a glass bottle made to hold a fixed volume of liquid. The neck of the bottle is provided with a finely-fitting ground glass stopper which, when pushed in, always occupies the same position. The stopper has a perforation running along its axis. When the bottle is filled with the liquid to the top of the neck and the stopper pushed in, the surplus liquid escapes through the perforation, leaving the bottle completely filled.

The specific gravity bottle may be employed for finding the specific gravity of a liquid or of a solid which is either in the form of a powder or in small fragments.



(I) *To determine the specific gravity of a given liquid.*

Let the bottle be weighed, firstly when empty, secondly when filled with water, and thirdly when filled with the given liquid; let these weights be $W_1$, $W_2$ and $W_3$ respectively.

We have then

the wt. of water filling the bottle $= W_2 - W_1$,

the wt. of the liquid filling the bottle $= W_3 - W_1$.

Hence, the sp. gr. of the given liquid

$$= \frac{W_3 - W_1}{W_2 - W_1}. \qquad \ldots \qquad (1)$$

(II) *To determine the specific gravity of a solid.*

After taking the experimental readings, let

$W_1 =$ wt. of the solid,

$W_2$ = wt. of the bottle when full of water,
$W_3$ = wt. of the bottle when it contains the solid and is filled up with water.

Then we get

$W_3 - W_2$ = wt. of the solid — wt. of water displaced by the solid.

∴ wt. of water displaced by the solid

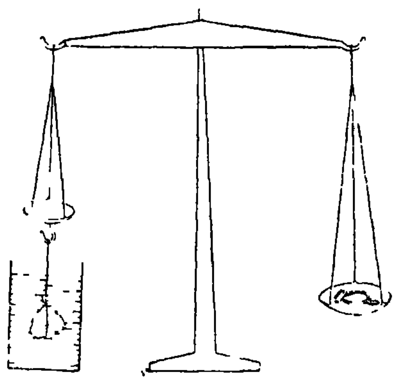
$$= W_1 - W_3 + W_2.$$

Hence, the sp. gr. of the solid

$$= \frac{W_1}{W_1 + W_2 - W_3}. \quad . \quad . \quad . \quad (2)$$

It must be noted that this method is applicable to only those solid substances which are insoluble in water. Moreover, the solid should be such as not to produce chemical changes nor float in water. For the case of a body, soluble in water, another liquid in which it is insoluble, should be used. By knowing the sp. gr of this new liquid [Vide I], the sp. gr. of the solid can be easily calculated.

**A·3. The Hydrostatic Balance.** This is an ordinary balance adapted to weighing bodies in liquids. One of its scale-pans is suspended by shorter chains or wire than the other. Attached to the bottom of the pan which is suspended by the shorter chain, there is a hook from which a body, while immersed in a liquid, may be suspended by a string or wire.



By means of a hydrostatic balance the sp. gr. of solids as well as liquids can be determined.

(I) *To determine the specific gravity of a solid heavier than water.*

Let

$W$ = wt. of the solid,

$W'$ = *apparent* wt. of the solid, when weighed in water.

Since the weight of the water displaced by the solid is $W - W'$, the sp. gr. of the solid

$$= \frac{W}{W - W'}. \quad . \quad . \quad . \quad . \quad (1)$$

(II) *To determine the specific gravity of a solid lighter than water.*

In this case the solid will not sink in water by itself; it must be attached to another heavy body, called a *sinker*, so that the two together may sink in water.

Let the following be found by experimental observations:—

$W_1$ = wt. of the solid,

$W_2$ = wt. of the sinker in water,

$W_3$ = wt. in water of the solid and sinker together.

Now

$W_3$ = wt. of the sinker in water + wt. of the solid − wt. of water displaced by the solid

$= W_2 + W_1$ − wt. of water displaced by the solid.

$\therefore$ wt. of water displaced by the solid

$$= W_2 + W_1 - W_3.$$

Hence, the sp. gr. of the solid

$$= \frac{W_1}{W_1 + W_2 - W_3}. \quad . \quad . \quad . \quad (2)$$

(III) *To determine the specific gravity of a given liquid.*

Take a piece of solid which will sink both in the given liquid and in water.

Let

$W_1$ = wt. of the solid,
$W_2$ = wt. of the solid in water,
$W_3$ = wt. of the solid in the given liquid.

Now

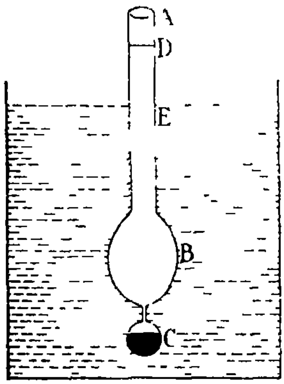
$W_1 - W_2$ = wt. of water displaced by the solid,
$W_1 - W_3$ = wt. of given liquid displaced by the solid.

Hence, the sp. gr. of the given liquid

$$= \frac{W_1 - W_3}{W_1 - W_2}. \quad \quad \quad (3)$$

**A·4. Hydrometers.** A hydrometer* consists essentially of a bulb and a uniform stem. The bulb is loaded at one end, so the hydrometer floats in liquids in a vertical position. Hydrometers are of very different forms, and may be used for finding the specific gravity of liquids or of solids. We shall describe here the two principal varieties, viz., the Common Hydrometer and Nicholson's Hydrometer.

**A·41. Common Hydrometer.** This hydrometer, generally made of glass, consists of a uniform straight stem ending in a bulb $B$; below $B$ is another small bulb $C$ loaded usually with mercury so that the instrument may float with the stem vertical. It is used to find the specific gravity of a liquid.




*To determine the specific gravity of a given liquid.*

Suppose the hydrometer sinks up to the marks $D$ and $E$ when immersed in water and in the

* This instrument is said to have been invented by Hypatia, the daughter of Theon Alexandrinus, who flourished about the end of the fourth century.

given liquid respectively. Let $V$ be the volume of the instrument and $\alpha$ the cross-sectional area of the stem. Then the volumes of water and the given liquid displaced by the hydrometer are $(V - \alpha \,.\, AD)$ and $(V - \alpha \,.\, AE)$ respectively.

Since in each case the weight of the displaced liquid is equal to the weight of the hydrometer, we have

$$(V - \alpha . AE) . s = V - \alpha . AD,$$

where $s$ is the sp. gr. of the given liquid.

$$\therefore \qquad s = \frac{V - \alpha . AD}{V - \alpha . AE} \qquad . \quad . \quad . \quad . \quad . \quad (1)$$

The weight of the liquid displaced, being equal to that of the instrument, is always the same; for this reason a common hydrometer is sometimes called the *constant weight* hydrometer.

**A·41. Graduation of the Common Hydrometer.** For ready use the stem of a common hydrometer is graduated with markings to indicate the sp. gravities of liquids in which the instrument would sink up to those markings.


To explain the principle of graduation, suppose the stem prolonged to $O$, so that the volume of $OA$ is equal to that of the hydrometer, that is

$$V = \alpha . OA.$$

From (1) of **A·41**, the sp. gr. $s$ of the liquid in which the hydrometer sinks up to the mark $E$ is given by

$$s = \frac{V - \alpha . AD}{V - \alpha . AE} = \frac{\alpha . AO - \alpha . AD}{\alpha . AO - \alpha . AE}$$

$$= \frac{OD}{OE} .$$

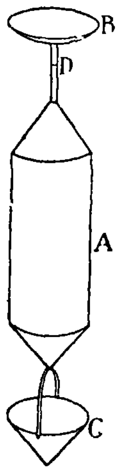
Hence, if $D$ be the point up to which the hydrometer sinks in water, so that $OD$ is a constant

for the instrument, the graduation $E$ corresponding to the sp. gr. $s$ of the liquid is given by

$$OE = \frac{OD}{s}. \quad \ldots \quad (1)$$

It follows from above that *if the values of the sp. gr. s vary in Arithmetical Progression, the corresponding distances OE of the graduation points would be in Harmonical Progression; conversely, if the distances OE are in A.P., the corresponding sp. gr. are in H. P.*

**A·42. Nicholson's Hydrometer.** The instrument consists of a hollow cylinder $A$, generally made of brass, with a thin stem supporting a pan or cup $B$ above it. Below $A$ there is another cup $C$ connected with $A$ by a wire and loaded sufficiently so that the instrument may float with its stem vertical.




This hydrometer is used for finding the specific gravity of solids as well as of liquids. It may also be used for weighing small bodies and for comparing the specific gravities of two liquids.

On the stem there is a well-defined mark $D$, and while using the hydrometer, the instrument is so weighted that it sinks in the liquid always exactly up to the mark $D$. Thus the amount of liquid displaced by the instrument is always the same, and for this reason this hydrometer is sometimes called the *constant volume* hydrometer.

(I) *To determine the specific gravity of a given liquid.*

Let the following experimental readings be taken:—

$W$ = wt. of the hydrometer,

$W_1$ = weight required to be put on the pan $B$ to sink the hydrometer in the given liquid up to the mark $D$,

$W_2$ = weight required to sink the hydrometer in water up to the mark $D$.

Then

$W + W_1$ = wt. of the liquid displaced by the hydrometer,

$W + W_2$ = wt. of water displaced by the hydrometer.

Hence, the sp. gr. of the liquid

$$= \frac{W + W_1}{W + W_2}. \quad . \quad . \quad . \quad . \quad (1)$$

(II) *To determine the specific gravity of a solid.*

Let

$W_1$ = wt. required to be put on the pan $B$ to sink the hyd. to $D$,

$W_2$ = wt. required to sink the hyd. to $D$, when the solid also is on the pan,

$W_3$ = wt. required to sink the hyd. to $D$, when the solid is placed in the cup $C$ underneath water.

Now

wt. of the solid $+ W_2$ = wt. of the solid in water $+ W_3$.

$\therefore \quad W_3 - W_2$ = wt. of the solid—wt. of the solid in water

= wt. of water displaced by the solid.

Also

$W_1 - W_2$ = wt. of the solid.

Hence, the sp. gr. of the solid

$$= \frac{W_1 - W_2}{W_3 - W_2}. \quad . \quad . \quad (2)$$

**A·5. Specific Gravity of Gases.** In determining the specific gravity of gases, air at 0°$C$. and at ordinary atmospheric pressure is taken as the standard substance. The process is attended with so many practical difficulties that many precautions are needed in order

to obtain accurate results. This subject is properly treated in books on *Heat* to which reference may be made, if required.

**A·6. Illustrative Examples.** (i) *A common hydrometer is placed successively in three kinds of liquids. It floats with* 2 *inches of its stem out of the first liquid, with* 4 *inches out of the second, and* 5 *inches out of the third; the sp. gr. of the first and second liquids being* 0·6 *and* 0·8 *respectively, find the sp. gr. of the third liquid.*

Let $V$ be the total volume of the hydrometer in cubic inches and $K$ sq. inch the cross-sectional area of its stem. Then the volumes of the three liquids displaced are $V-2K$, $V-4K$ and $V-5K$ respectively.

If $W$ denotes the weight of the hydrometer, $w$ that of a cubic inch of water and $x$ the sp. gr. of the third liquid, then we have

$$(V-2K)\times 0\cdot 6w=(V-4K)\times 0\cdot 8w=(V-5K)\times xw. \qquad (1)$$

From the first two of (1), we get

$$6V-12K=8V-32K,$$

or

$$V=10K. \qquad (2)$$

Again

$$6V-12K=10xV-50xK,$$

or

$$V(6-10x)=K(12-50x),$$

or

$$10K(6-10x)=K(12-50x),$$

or

$$50x=48,$$

or

$$x=24/25=0\cdot 96.$$

(ii) *The weight of a Nicholson's hydrometer is* 7 *ozs. When it is placed in a liquid, a weight of* 3 *ozs. is needed to be put on the pan to make it sink to the fixed mark, while a weight of* 4 *ozs. is required to make it sink to the same mark in another liquid. Compare the specific gravities of the two liquids.*

Let $V$ be the volume of the hydrometer excluding the portion above the fixed mark. If $s_1$, $s_2$ be the sp. gr. of the two liquids, then we have

$$(7+3) \text{ ozs.} = Vs_1,$$

and

$$(7+4) \text{ ozs.} = Vs_2.$$

Hence

$$s_1 : s_2 = 10 : 11.$$

## Examples A

1. The whole volume of a common hydrometer is 15 c. cm. and its stem is 3 mm. in diameter. The hydrometer floats in a liquid $A$ with 3 cm. of the stem above the surface, and in another liquid B with 6 cm. above the surface. Compare the densities of the liquids. [*Madras*, 1936]

2. A Nicholson's hydrometer, of weight $4\frac{3}{4}$ ozs. requires weights of 2 ozs. and $2\frac{5}{8}$ ozs. respectively to sink to the fixed mark in two different liquids. Compare the specific gravities of the two liquids.

3. A piece of metal which weighs 15 ozs. in water is attached to a piece of wood which weighs 20 ozs. in vacuum, and the two together weigh 10 ozs. in water Find the sp gr of the wood [*Calcutta*, 1913]

4. A specific gravity bottle, full of water, weighs 44 gr. and when some pieces of iron, weighing 10 gr. in air, are introduced into the bottle and the bottle is again filled up with water, the combined weight is 52 7 gr. Find the sp. gr. of iron.

5. The effect of air being neglected, the sp. gr. of a solid body is found by a specific gravity bottle to be $\sigma$, if $a$ be the sp. gr. of air, show that the real specific gravity of the body is $\sigma - a(\sigma - 1)$.

6. A solid is placed in the upper cup of a Nicholson's hydrometer and it is then found that 5 ozs are required to sink the instrument to the fixed point. When the solid is placed in the lower cup, 7 ozs. are required; and when the solid is taken away altogether, 10 ozs. are required What is the specific gravity of the solid? [*Calcutta*, 1915]

7. A common hydrometer has a small portion of its bulb rubbed off from frequent use. In consequence when placed in the water it appears to indicate the specific gravity of water as 1·002, find what fraction of its weight has been lost

8. A hydrometer indicates graduations $a, b, c$, in liquids whose densities are $\rho_1, \rho_2, \rho_3$, respectively. Show that

$$\frac{b-c}{\rho_1} + \frac{c-a}{\rho_2} + \frac{a-b}{\rho_3} = 0$$ [*Patna*, 1933]

9. A common hydrometer sinks to the points $A$, $B$, $C$ in

liquids whose densities are $\varrho_1$, $\varrho_2$, $\varrho_3$ respectively. If $AB = a$, $BC = b$, and $AC = a + b$, prove that

$$\frac{b}{\varrho_1} + \frac{a}{\varrho_3} = \frac{a+b}{\varrho_2}.$$ [*Bombay*, 1937]

10. The specific gravity of a body found by Nicholson's hydrometer is $\sigma$ when the effect of the air is neglected; prove that the real specific gravity is $\sigma - \alpha(\sigma - 1)$, where $\alpha$ is the sp. gr. of the air. Also if $W$ be the apparent weight of the body as found from the experiment, find its real weight.

11. A common hydrometer whose volume is $V$ and the cross-section of whose stem is $V/c$, has lengths $a$ and $b$ of its stem uncovered when floating in one or the other of two liquids. In a mixture of weights of the two liquids in the proportion $m : 1$, $x$ is uncovered, and in a mixture of volumes in the proportion $m : 1$, $y$ is uncovered. Prove that $$(x - y)(c - y) = (a - y)(b - y)$$

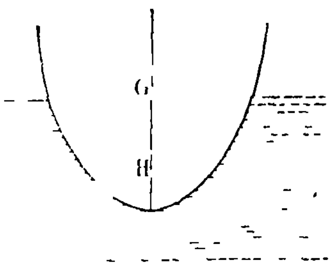
12. A common hydrometer is used to determine the sp. gr. of a liquid which is at a temperature higher than that of water. When the hydrometer is transferred from water to the liquid the specific gravity appears at first to be $\sigma$, but afterwards to be $\sigma'$. Show that the true sp. gr. at the temperature of the water is $\sigma + (\alpha'/\alpha)(\sigma' - \sigma)$, where $\alpha$ and $\alpha'$ are the coefficients of expansion of the hydrometer and the fluid respectively. [*Benares*, 1937]

# METACENTRE AND STABILITY OF FLOATING BODIES

**B·1. Stable and Unstable Equilibrium.** If a body, which is in equilibrium under certain forces, be slightly displaced from its position of equilibrium, it may be that the forces acting on it tend to restore it to its original position, or it may be that they tend to take it farther away from the original position. In the first case the body is said to be in *stable equilibrium*, while if second be the case, it is in *unstable equilibrium*.

There is another case which arises when the body is in equilibrium in the displaced position as well, i.e., it remains in equilibrium in any position; then the body is said to be in *neutral equilibrium*.

When a body floats in water, or in any other liquid, we have seen in 5·2 that




(i) the weight of the body is equal to the weight of the liquid displaced, and

(ii) the centre of gravity $G$ of the body and the centre of buoyancy $H$ are in the same vertical line.

A small displacement given to this body may be resolved into simpler displacements of translation and rotation.

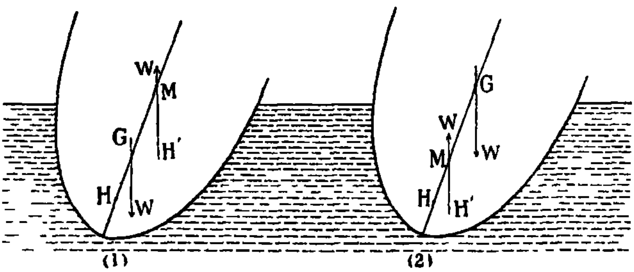
Firstly, a small *horizontal displacement* without rotation obviously does not affect the equilibrium in any way, for

there is no resultant horizontal force on the body. For such displacements, therefore, the equilibrium is *neutral.*

Secondly, a small *vertical displacement* downwards increases the force of buoyancy which tends to restore the body to its former position, and a small upward vertical displacement decreases buoyancy which again tends to bring the body back to its original position. For such displacements, therefore, the body is in *stable* equilibrium.

Lastly remains to be considered the case when the body is given a small *rotational displacement* which changes the shape and not the mass of the displaced liquid. In what follows we proceed to consider this case.

**B·2. Metacentre.** Let $W$ be the weight of a floating body, $G$ its centre of gravity and $H$ the centre of buoyancy. Suppose the body undergoes a small rotational displacement about a horizontal axis and suppose that this rotation does not alter the mass of the liquid displaced. Let $H'$ be the centre of buoyancy in the displaced position. Since the mass of the liquid displaced remains unaltered, the force of buoyancy is the same as before, namely, equal to the weight of the body.




The forces acting on the body in the displaced position are (i) the weight, $W$, of the body acting vertically downwards through $G$ and (ii) the force of buoyancy, $W$, acting vertically upwards through $H'$.

The vertical through $H'$ will intersect $HG$ if the displacement takes place in a plane of symmetry of the body. Supposing such to be the case, let the vertical through $H'$ meet $HG$ in $M$.

In Fig. (1), $M$ is above $G$, and evidently the two forces forming a couple, tend to bring the body back to its original position. The equilibrium in this case is therefore *stable.*

In Fig. (2), $M$ is below $G$, and the forces forming the couple tend to make the body recede further away from its original position. The equilibrium in this case is therefore *unstable.*

The point $M$ is called the **metacentre** and $GM$ the **metacentric height** of the floating body corresponding to the particular displacement made.

We conclude from above that *a floating body is in stable or unstable equilibrium according as the metacentre is above or below the centre of gravity of the body.*

*If the metacentre coincides with the centre of gravity, the equilibrium is neutral.*

To express concisely,

$$\textbf{the equilibrium is } \left.\begin{array}{r}\textbf{stable}\\ \textbf{neutral}\\ \textbf{unstable}\end{array}\right\} \textbf{according as HM } \left.\begin{array}{c}>\\ =\\ <\end{array}\right\} \textbf{HG.}$$

**B·21. Definitions.** If a body be floating partially immersed in a liquid in any position, the section of the body made by the plane of the surface of the liquid is termed the corresponding **Plane of Floatation.**

If a body floating in a homogeneous liquid be supposed to take in turn every possible position for which the volume of the liquid displaced remains constant, the locus of the centre of buoyancy is called the **Surface of Buoyancy.**

**B·3. Determination of the Metacentre.** The position of the metacentre depends in any case on the surface in contact with the liquid, i.e., on the shape of the body itself. Its formal determination involves certain ideas which are beyond the scope of this book. It may, however, be assumed at this stage without proof that *the distance between the centre of buoyancy H in the position of equilibrium and the metacentre M of the body is given by the formula*

$$\mathrm{HM} = \frac{\mathrm{Ak}^2}{\mathrm{V}},$$

*where k is the radius of gyration* of the area A of the section in the plane of floatation, and V is the volume immersed.*

In some simple cases, however, metacentre can be found more easily. Thus for a spherical surface in contact with the liquid, the metacentre is always the centre of the sphere of which the surface is a part, because the force of buoyancy for such a surface always passes through this centre. In the same way, if the immersed surface is the curved surface of a circular cylinder whose axis is horizontal, the metacentre will lie on the axis of the cylinder, for the thrust on each element of the surface intersects the axis.

**B·4. Illustrative Examples.** (i) *A solid and homogeneous body consists of a cylinder joined to a hemisphere on the same base and floats*

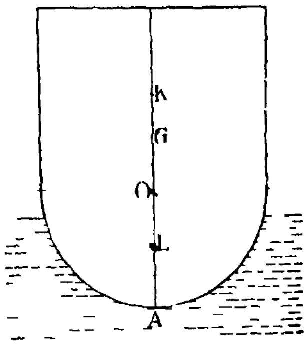
* The *radius of gyration* of an area is defined in terms of its *moment of inertia* about a given axis of rotation. If $x$ be the distance from the axis of rotation of an element $dA$ of the area $A$, the moment of inertia is

$$\int_{(A)} x^2\, dA,$$

where the integration extends over the whole area. If the value of the moment of inertia be denoted by $Ak^2$, then $k$ is called the radius of gyration.

The value of $k$ in simple cases of common occurrence is given in the Appendix C.

*with the hemispherical portion partly immersed in water. Find the greatest height of the cylinder consistent with stability.* [Nagpur, 1939]




Let $a$ be the radius of the hemisphere whose centre is $O$, and let $h$ be the height of the cylinder.

The weight of the hemisphere is $\frac{2}{3}\pi a^3 \rho g$ acting through $L$, where $OL = 3a/8$, and $\rho$ is the density of the body. Also the weight of the cylindrical part is $\pi a^2 h \rho g$ and acts through $K$, where $OK = h/2$. Therefore the height of the centre of gravity $G$ of the body above the lowest point $A$ is

$$AG = \frac{\frac{2}{3}\pi a^3 \rho g(a - 3a/8) + \pi a^2 h \rho g(a + h/2)}{\frac{2}{3}\pi a^3 \rho g + \pi a^2 h \rho g}$$

$$= \frac{5a^2/4 + 3h(a + h/2)}{2a + 3h}.$$

Now the metacentre is at the centre $O$ of the spherical part, its height $OA$ being $a$.

Hence for stability, we must have

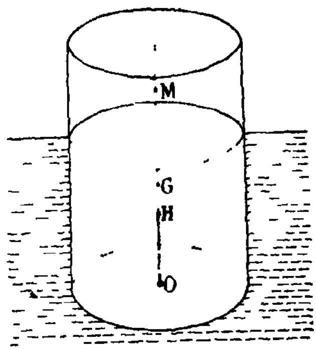
$$AG \leqslant AO,$$

or $$5a^2/4 + 3h(a + h/2) \leqslant a(2a + 3h),$$

or $$3h^2/2 \leqslant 3a^2/4,$$

or $$h \leqslant \sqrt{2}\,a/2.$$

(ii) *Discuss the stability of a uniform right circular cylinder of length h, radius a, and density $\rho$, which floats in a liquid of density $\sigma$, with its axis (i) vertical, (ii) horizontal.*




(*i*) When the axis is vertical, let a length $h'$ of the axis be immersed. Then

$$\sigma h' = \rho h. \quad . \quad . \quad . \quad . \quad (1)$$

Let O be the centre of the base, $H$ the centre of buoyancy, $G$ the centre of gravity and $M$ the metacentre.

Then $OH = \frac{1}{2}h'$, $OG = \frac{1}{2}h$.

The surface section is a circle of radius $a$, so that $k = a/2$. Hence

$$Ak^2 = \pi a^2 . a^2/4 = \tfrac{1}{4}\pi a^4.$$

Also the volume immersed $V = \pi a^2 b'$.

$$\therefore \qquad HM = Ak^2/V = \tfrac{1}{4}\, a^2/b'.$$

For stability, we must have

$$OM > OG,$$

or $$OH + HM > OG,$$

i.e. $$\tfrac{1}{2}b' + \tfrac{1}{4}a^2/b' > \tfrac{1}{2}b,$$

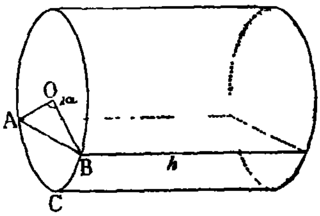
or $$a^2 > 2b'(b - b'),$$

or $$a^2 > 2(\rho b/\sigma)(b - \rho b/\sigma), \quad \text{from (1)},$$

or $$a^2/b^2 > 2\rho(1 - \rho/\sigma)/\sigma,$$

which is the required condition.

(*ii*) When the axis is horizontal, the equilibrium is neutral for rolling displacements. The metacentre $M_1$ in this case lies on the axis.




The section by the water surface is a rectangle, one of whose sides is $h$ and the other is $AB$ equal to $b$, say. If $2\alpha$ be the angle subtended by the side $b$ at the centre O of one end-face, then

$$\sin\alpha = b/2a. \qquad \ldots\ldots\ldots \quad (1)$$

Also the area of the segment $ACB$ immersed in the liquid is

$$a^2(\alpha - \sin\alpha\cos\alpha).$$

$$\therefore \qquad a^2(\alpha - \sin\alpha\cos\alpha)h\sigma = \pi a^2 h\rho,$$

or $$\sigma(\alpha - \sin\alpha\cos\alpha) = \pi\rho, \qquad \ldots\ldots \quad (2)$$

which determines $\alpha$, and (1) then gives $b$.

The moment of inertia $Ak^2$ in tilting displacement will be greater than in rolling if $h > b$. If this condition be satisfied, the metacentre will be higher than $M_1$, and the equilibrium will be stable.

## Examples B

1. A solid cylinder floats in a liquid with its axis vertical. If the ratio of the sp. gr. of the cylinder to that of the fluid be $\sigma$, prove that the equilibrium is stable if the ratio of the radius of the base to the height be greater than $\sqrt{2\sigma(1-\sigma)}$.

2. Show that in the case of a right circular cylinder of radius $a$ and height $h$, floating with its axis vertical in any liquid, the equilibrium will be stable whatever be the sp. gr., if $\sqrt{2}\,a > h$.

[*Nagpur*, 1942, 1943]

3. A solid body consists of a right cone joined to a hemisphere on the same base and floats with the spherical portion partly immersed. Prove that the greatest height of the cone consistent with stability is $\sqrt{3}$ times the radius of the base. [*Benares*, 1943]

4. A right prism whose base is an equilateral triangle, floats in water with the lateral edges horizontal and one of them below the surface. Show that the equilibrium is stable for all displacements into which the lateral edges remain horizontal, if it be given that the sp. gr. of the prism $> 9/16$. [*Benares*, 1940]

5. A solid cone, of semi-vertical angle $\alpha$, height $h$ and sp. gr. $\sigma$, floats in equilibrium in a liquid of sp. gr. $\varrho$, with its axis vertical. Determine the condition for which the equilibrium is stable. [*Nagpur*, 1941]

6. A uniform right circular cone of semi-vertical angle $30°$, is floating with its axis vertical and vertex downwards in a liquid whose density is $3/2$ its own. Determine whether the equilibrium is stable or unstable. [*Nagpur*, 1932]

7. A thin uniform lamina whose shape is that of an isosceles triangle, floats in water with its plane vertical and the base horizontal and above the surface. Show that the equilibrium is stable, if

$$\sigma > \cos^4\alpha,$$

where $\sigma$ is the sp. gr. of the lamina and $2\alpha$ the vertical angle.

8. Show that when a uniform hemisphere of density $\varrho$ and radius $a$ floats with its plane base immersed in a homogeneous liquid of density $\sigma$, the equilibrium is stable and the metacentric height is $\frac{3}{8}a(\sigma - \varrho)/\varrho$. [*M.T.*]

9. If a segment of a sphere of density $\sigma$ floats in a liquid of density $\varrho$, prove that the position in which the plane face is immersed and horizontal, is a position of stable equilibrium and that the metacentric height is $(\varrho - \sigma)/\sigma$ times the distance of the centre of gravity of the segment from the centre of the sphere.

10. Prove that a circular cylinder of radius $a$ and length $a/n$ cannot float upright in water in stable equilibrium if its sp. gr. lies between

$$\tfrac{1}{2}[1 - \sqrt{1 - 2n^2}] \text{ and } \tfrac{1}{2}[1 + \sqrt{1 - 2n^2}].$$

What will happen if $2n^2 > 1$?

11. Show that a homogeneous right circular cone of vertical angle $2\alpha$ cannot float in stability with the axis vertical and vertex upwards, unless its density as compared with that of the liquid, is less than $1 - \cos^6\alpha$.

12. The paraboloid of revolution, $x^2 + y^2 = 2cz$, floats with its axis vertical and vertex downwards. Show that the distance between the centre of buoyancy and the metacentre is independent of the length of the axis immersed.

13. A cylindrical cup is formed of thin sheet metal, the height being twice the diameter; the surface density of the plane bottom is $n$ times that of the curved surface, and the weight of the cup is half that of the water which would fill it. Show that the cup will float in stable equilibrium with its generator vertical, if $n > 56/9$. [*M. T.*]

## APPENDIX C

# SOME USEFUL GEOMETRICAL RESULTS

**Circle.** If $r$ be the radius of the circle, the circumference $= \mathbf{2\pi r}$, and its area $= \boldsymbol{\pi r^2}$.

**Cylinder.** The area of the curved surface of a cylinder of height $h$ and base radius $r$, $= \mathbf{2\pi rh}$, and its volume $= \boldsymbol{\pi r^2 h}$.

**Sphere.** For a sphere of radius $r$,

the area of the surface $= \boldsymbol{4\pi r^2}$,

and its volume $= \boldsymbol{\tfrac{4}{3}\pi r^3}$.

The area of the zone of a sphere (i.e., of the surface of the sphere intercepted between two parallel planes)

= circumference of the sphere × perpendicular distance between the planes

$= \mathbf{2\pi rd}$.

The volume of the portion of a sphere of radius $r$ included between two parallel planes at distances $d_1$ and $d_2$ from the centre

$$= \tfrac{1}{3}\pi(\mathbf{d_2} - \mathbf{d_1})\{3\mathbf{r}^2 - (\mathbf{d_1}^2 + \mathbf{d_1 d_2} + \mathbf{d_2}^2)\}.$$

**Cone.** The area of the curved surface of a cone of height $h$, semi-vertical angle $\alpha$ and base radius $r$

= ½ slant side × perimeter of the base

$= \boldsymbol{\pi r\sqrt{h^2 + r^2} = \pi h^2.}$ **tan** $\alpha$ **sec** $\alpha$.

Its volume = ⅓ Height × Area of the base

$= \boldsymbol{\tfrac{1}{3}\pi r^2 h}$.

The volume of a frustum of a cone

$$= \tfrac{1}{3}\pi \mathbf{d}(\mathbf{r_1}^2 + \mathbf{r_1 r_2} + \mathbf{r_2}^2),$$

where $r_1$, $r_2$ are the radii of its circular ends, and $d$ is the perpendicular distance between them.

**Paraboloid of revolution.** This is a solid formed by revolving a parabola about its axis.

The volume of a portion of it cut off by a plane perpendicular to its axis

= half the volume of the cylinder on the same plane base and of the same height

= **½ area of its plane base × its height.**

**Centre of Gravity.**

(1) The C. G. of a circular arc of radius $r$ subtending an angle $2\alpha$ at the centre is a point on the bisector of the angle $2\alpha$, whose distance from the centre

$$= r \cdot \frac{\sin \alpha}{\alpha}.$$

For a semi-circle (i.e., when $\alpha = \pi/2$), this distance of the C. G.

$$= 2r/\pi.$$

(2) The C. G. of a sector of a circle of radius $r$ subtending an angle $2\alpha$ at the centre is a point on the bisector of the angle $2\alpha$, whose distance from the centre

$$= \tfrac{2}{3}\, r . \frac{\sin \alpha}{\alpha}.$$

For a semi-circle, this distance of the C. G.

$$= \frac{4r}{3\pi}.$$

(3) The C. G. of a hemisphere of radius $r$, is a point on the radius normal to the base, whose distance from the centre $= \tfrac{3}{8}r$.

For the curved surface of the hemisphere, this distance of the C. G. from the centre $= \tfrac{1}{2}r$.

(4) The C. G. of a right circular cone of height $h$, is a point on the axis whose distance from the vertex $= \tfrac{3}{4}h$.

For the curved surface of the cone, the distance of the C. G. from the vertex $= \tfrac{2}{3}h$.

The distance from the axis of the C. G. of the half-cone

$$= r/\pi.$$

(5) The C. G. of a semi-ellipse bounded by the major axis is a point on the minor axis whose distance from the centre

$$= 4b/3\pi,$$

where $b$ is the length of the semi-minor axis.

**Radius of Gyration.** The value of $k^2$, where $k$ is the radius of gyration in the expression for *moment of inertia*, viz., $Ak^2$, is given below for some simple cases.

(1) For a thin rod of length $l$ about an axis through the middle point perpendicular to the rod,

$$k^2 = l^2/12.$$

(2) For a rectangular lamina, whose sides are $a$ and $b$, about a straight line through its centre parallel to the side $b$,

$$k^2 = a^2/12.$$

(3) For a circular area of radius $a$, about a diameter,

$$\mathbf{k}^2 = \mathbf{a}^2/4.$$

(4) For an elliptic area whose semi-axes are $a$ and $b$,

$$\mathbf{k}^2 = a^2/4 \text{ about the axis } 2b, \text{ and}$$

$$\mathbf{k}^2 = b^2/4 \text{ about the axis } 2a.$$

(5) For a triangular area $ABC$ about a straight line through $A$ in the plane of the area,

$$\mathbf{k}^2 = \tfrac{1}{6}(\beta^2 + \beta\gamma + \gamma^2),$$

where $\beta$ and $\gamma$ are the distances of $B$ and $C$ from the axis of rotation.

# ANSWERS TO THE EXAMPLES

## Examples I (Pages 16-17)

**1.** 156·25 k. gr. **2.** $41\frac{2}{3}$ lbs. wt. **3.** 180 lbs. wt.
**4.** 560 lbs. wt. **5.** 1 : 7. **7.** 6·48 cu. ft.
**8.** 16·383 nearly. **9.** 147 : 250. **10.** $(vs+v's')/\sigma$
**11.** 1. **12.** 6, 2.

## Examples II (Pages 29-31)

**1.** $36\frac{13}{17}$ metres. **2.** 15·36 lbs., 23·68 lbs.
**3.** 46·08 ft. **4.** 69·12 ft. **5.** 98 ft.
**9.** 1443·6, 106·15. **10.** $\frac{1}{2}n(n+1)g\varrho b$, where $b$ is the thickness of each stratum.
**12.** 29·92 inches.

## Examples III (Pages 40-44)

**1.** 22500 lbs. wt **2.** 3373·4375 lbs. wt.
**3.** 127·12 tons wt. **4.** $1001953\frac{1}{8}$ lbs. wt.
**5.** 1875 lbs. wt. **6.** 531·25 lbs. wt.
**7.** 35·148 lbs. wt. **8.** $2\frac{29}{44}$ lbs. wt.
**9.** $\frac{2}{3}b(a^2-b^2)w$. **10.** 14000 lbs. wt., $4\sqrt{2}$ ft.
**11.** 11·48 ozs. **12.** $1 : \sqrt[3]{2}-1$.
**13.** The line divides a side in the ratio $1 : \sqrt{2}-1$.
**14.** $CE=(3/4).AB$.
**15.** $AE$ is the line, where $E$ lies in $CD$ such that $CE=CD/4$.
**16.** $\sqrt{2}+1 : 1$. **17.** 9 ft. **21.** $4(1+\sqrt{10})/3$ inches.
**22.** $\{1/(1+\varrho)\}\{(2a-b)^2+b(1+\varrho)(2a-b)\}^{1/2}-(2a-b)/(1+\varrho)$.
**24.** The depths of the successive dividing lines are $b\sqrt{1/n}, b\sqrt{2/n},\ldots,$ $b$ being the ht. of the parallelogram.
**25.** Divide the horizontal diameters into $n$ equal parts; the ordinates at these points will divide the arc of the semi-circle in the required points.
**26.** $2W/\sqrt{3}$, $W/\sqrt{3}$, where $W$ is the wt. of water in the cube.
**27.** $18\frac{9}{113}$. **30.** $4\pi c^3w/3$.
**31.** The plane cuts off $\sqrt[3]{1/2}$ of the axis. **34.** $3041\frac{2}{3}$ lbs. wt.

## Examples IV (Pages 53-54)

1. $250\pi w$.

4. $hr^2(1-\pi/6)w$, $hr^2(1+\pi/6)w$, $h$ and $r$ being ht. and radius of the cone.
5. $\frac{1}{3}\pi h(a^2+ab-2b^2)w$, $28\frac{3}{4}\frac{1}{8}$ lbs. wt.

6. $hr^2(2+\pi/2)w$, $h(\pi r^2/2-2r^2+2rd)w$.
8. $250\pi w/3$.

11. $\sqrt{6}a^3w/12$, $\sqrt{2}a^3w/3$.

## Examples V (Pages 61-63)

2. $r^2hw$, $\frac{1}{2}\pi r^2w(r+4r/3\pi)$, $r$ and $h$ being the radius and ht. of the cone.
3. $2r^2hw$, $r$ and $h$ being the radius and ht. of the cylinder.
4. $whr\sqrt{(h^2+\pi^2r^2/4)}$, $\tan^{-1}(\pi r/2h)$ with the horizontal.
5. $\frac{1}{2}wr^2h\sqrt{(\pi^2+16)}$, $\tan^{-1}(\pi/4)$ with the horizontal.
7. $\frac{2}{3}a^3w$.
9. $5\pi a^3w/6$ acting at an angle $\tan^{-1}(3/4)$ to the vertical.
11. $\frac{1}{3}ahw\sqrt{(\pi^2a^2+4h^2)}$, $a$ and $h$ being the radius and ht. of the cone.
15. $120^\circ$.

## Examples VI (Pages 68-70)

1. $\tan^{-1}(1/3)$ with the horizontal, $178.88w$ grains.
3. $\pi a^3w/\sqrt{2}$, $\pi a^2(h+a)w\sqrt{2}$, $a$ and $h$ being the base radius and ht. of the cylinder.
4. $\frac{2}{3}\pi a^2w\sqrt{(13-12\cos\theta)}$, $\frac{2}{3}\pi a^3w\sqrt{(13+12\cos\theta)}$ through the centre, $\theta$ being the inclination of the diam. plane to the horizontal.
5. $2a^3w/3$.

## Examples VII (Pages 83-85)

2. $38087\cdot5$ lbs., $6\cdot4$ ft.
3. 9 grammes.

6. $2(a^3-b^3)/(a^2-b^2)$.
8. Divides the diagonal in the ratio 7 : 5.

## Examples VIII (Pages 89-90)

1. Depth of C. P. is $11h/8$, where $h$ is the depth of the base.
2. $\frac{1}{4}(hH)/(2h+3H)$, where $h$ is the ht. of the triangle.
6. $\frac{1}{2}h(\sqrt{5}-1)$ is the depth of the line.

## Examples IX (Pages 106-109)

1.* $5\pi a^2w/4$.
9. $3\pi a/16$ below the centre.

10. $a^2/(4b)$ below the centre, $a$ being the semi-major axis.

11. $16a^{1/2}b^{5/2}w$, $4b/7$.
12. $32/(15\pi)$.

15. $26\sqrt{3}a/45$, $a$ being a side. 19. $9\pi a/16(3\pi+1)$.
22. $\bar{x}=4a/5, \bar{y}=16a/33$.
23. $\bar{x}=3a/28, \bar{y}=5a/14$, $x$-axis being in the surface.

## Examples X (Pages 115—117)

1. $19/25$. 2. $55/514$. 3. $2/3$, 36 c.cm.
4. 89·8 ft. 5. 0·48 cu. in. 6. 2·34 cm., 2·66 cm.
7. 0·8. 8. 93·75 cm.
10. $\frac{1}{4}$th of the cylinder is in the upper liquid.
13. (i) $h'=(\sigma/\varrho)^{1/3}h$, (ii) $h'=[1-(1-\sigma/\varrho)^{1/3}]h$, where $h$ is the ht. of the cone, $h'$ the ht. immersed and $\varrho$ the sp. gr. of the liquid.
14. $\sqrt[3]{1/2}$. 16. $104/209$ cu. ft.
18. $540w$, 900 cu. in. 19. 6500 tons.
22. $3/10$ and $13/16$ of its. vol. 23. $26\times10^{-5}$ of its vol.
24. $h/4-a/2$, $h/2-a$.

## Examples XI (Pages 120—122)

1. 387 : 466. 2. 1862·45 : 1869.
8. 1 c. c., 21·6 gr. per c. c. 4. 23 : 37.
5. 0·6, 3. 13. 31% by volume.

## Examples XII (Pages 125—127)

2. $2g/15$. 3. $g/14$. 6. 2 ft., 30°.
8. $\tan^{-1}(3/8)$, $2\pi(\varrho-\sigma)a^3/3$. 11. $27/32$.

## Examples XIV (Pages 145—147)

1. 26000 ft. 3. 33443·20 metres.
4. 3·24 gr. 5. 0·00013 cu. in.
6. $21\frac{7}{11}$ ft. 8. $310\frac{170}{283}$ c.c. 9. 429 : 224.
10. 0·2533 gr. 11. $37\frac{960}{5168}$ in. 12. 39·4°.
13. 28 ft. of water approx. 15. $r'(1+at):r(1+at')$.
16. $160\frac{2260}{2639}$ lbs.
17. The positive root of $x^2+(a+h)x-(h-a)h=0$, $h$ being the ht. of water barometer.
21. The piston divides the cylinder in the ratio 3 : 1.
23. $(T/V)(pv/t+p'v'/t')$.

## Examples XV (Pages 155—157)

1. 7 in. 2. 50 2/3 cm. 3. 5 in.
4. 27·146 in. 5. 30·5 in. 6. $27\frac{1}{3}$ in.
10. 3336 ft. 12. 8100 ft.

## Examples XVI (Pages 165—168)

1. 17 ft. 2. 176·8 cu. ft. 3. 27·1637 ft.
4. 1·155 $V$, where $V$ is the capacity of the bell.
5. 100 ft., $3\frac{3}{17}$ times the vol. of the bell. 6. $22\frac{8}{11}$ cu. ft.
8. $63\frac{1}{4}$ ft. 16. It remains constant.

## Examples XVII (Pages 176—177)

1. 4 ft. 2. 8·2 ft. nearly. 4. 2 ft., 9·37 ft.
5. 16 ft. 6. $8680\frac{5}{9}$ lbs. wt. 7. $nAl/B$ per min.
8. $625\pi/8$, $3125\pi/8$ lbs. wt. 9. 33 ft. 5 in.
11. At the end of 5th stroke.

## Examples XVIII (Pages 183—184)

1. 0·422 atmos. 2. 17·36 in. 3. 4 strokes.
4. 9 strokes. 5. 2465/6561.
10. 30 lbs. per sq. in., 33 strokes.
12. $\{(n+1)^{n+1}-n^{n+1}\}/\{(n+1)^n-n^n\}$.

## Miscellaneous Examples (Pages 185—193)

6. $\pi/2$. 9. $128w/5$, ($w$ = wt. of 1 cu. in. of water).
13. $\frac{1}{6}g\rho a^3(\pi^2-4\pi+8)^{1/2}$.
15. $\pi g\rho h^3\tan^2\alpha\sqrt{(1/9+\tan^2\alpha)}$, $\tan^{-1}(\cot\alpha/3)$ to the horizontal; (i) $2\cot^{-1}\sqrt{15}$, (ii) $2\cot^{-1}(\sqrt{3})$, $h$ being the ht. and $\alpha$ the semi-vertical angle of the cone.
16. $\frac{1}{3}\pi g\rho l^3\sin^2\alpha\cos\alpha\sqrt{1+3\sin^2\alpha}$.
18. Depth of C. P. is $3(\alpha^4-\beta^4)/5(\alpha^3-\beta^3)$. 23. $32/15\pi$.
24. $\bar{x}=5b/9, \bar{y}=(7/32).(b^3/a)^{1/2}$.
25. $\bar{x}=16a/15\pi, \bar{y}=32a/15\pi$, $x$-axis being in the surface and $a$ the radius.
43. (i) The ht. of the column falls to 24 inches, (ii) falls to $23\frac{1}{3}$ inches.

## Examples A (Pages 203—204)

1. $36/43$.    2. $18/19$.    3. $0 \cdot 8$.

4. $7\frac{9}{13}$.    6. $2\frac{1}{2}$.

7. $(1/501).[\sigma/(\sigma - 1)]$, $\sigma$ being the sp. gr. of the substance of the bulb.

10. $W(1 - a/\beta)[1 + a/(\sigma - \sigma a)]$, $\beta$ being the sp. gr. of the "weights."

## Examples B (Pages 210—212)

5. $\sigma/\varrho > \cos^6 a$.    6. Stable.